Manolescu's Seiberg-Witten Floer homotopy type

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Seiberg-Witten Floer homotopy type

Introduction

Morse homology

$\left.\begin{array}{l} M\text{: manifold,} \\ f: M \to \mathbb{R}\text{, Morse function} \end{array}\right\} \to H^*(M).$

Floer homology

∞-dim. Morse homology

- Gauge theory
  - Chern-Simons functional $CS: \mathcal{A} \to \mathbb{R}$
    $\to$ Instanton homology $HF(Y)$
  - Chern-Simons-Dirac functional $CSD: \mathcal{C} \to \mathbb{R}$
    $\to$ Seiberg-Witten Floer homology $HF^{SW}(Y)$
- Symplectic $\to$ Hamiltonian, Lagrangian
- Heegaard Floer homology

- Finite dim. Morse theory
  Morse function $\to$ CW complex structure of M.
- Floer theory
  $\to$ What is the underlying topological structure?
  - [Fukaya]... $\to$ Morse homotopy
  - [Cohen-Jones-Segal] $\to$ Floer homotopy type

[Manolescu]

In the Seiberg-Witten Floer case, (Y: 3-manifold with $b_1 = 0$ or 1,) it is defined a pointed S^1-space (prespectrum) $SWF(Y)$ s.t.

$$H_*(SWF(Y)) \cong HF_*^{SW}(Y).$$

Idea

- Gauge group $= \mathrm{U}(1)$.
- The compactness of the moduli.

} → Finite dimensional approximation

→ The Conley index

Cf. [Frauenfelder '04] → Moment Floer homology.

[Cohen '07] → Hamiltonian Floer homology of the cotangent bundle

References

1. [Manolescu1]
   Seiberg-Witten-Floer stable homotopy type of three-manifolds with $b_1 = 0$,
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2. [Manolescu2]
   A gluing formula for the relative Bauer-Furuta invariants,
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3. [Kronheimer-Manolescu]
   Periodic Floer pro-spectra for the Seiberg-Witten equations,
   preprint, arXive math/0203243.

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- Seiberg-Witten trajectories
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- Construction of the invariants

Seiberg-Witten trajectories

- Y: oriented closed 3-manifold, g: metric.
- c: Spin^c-structure.
  $\longrightarrow W_0$: the spinor bundle, $L = \det W_0$.
- If $b_1 = 0$, $\Rightarrow \exists$ flat connection A_0 on L unique up to gauge.
  $\longrightarrow \partial_0 \colon \Gamma(W_0) \to \Gamma(W_0)$, Dirac operator.
- $\mathcal{A}(L) := \{\mathrm{U}(1)\text{-connections on } L\} = A_0 + i\Omega^1(Y)$.
- For $A = A_0 + a$
  $\longrightarrow \partial_a = \rho(a) + \partial_0$, the Dirac op. associated to A,
  where $\rho(a)$ is the Clifford multiplication.

Seiberg-Witten Floer homotopy type
└Seiberg-Witten Floer stable homotopy types
└Seiberg-Witten trajectories

- $\mathcal{G} = \mathrm{Map}(Y, S^1) \curvearrowright \mathcal{C} := i\Omega^1(Y) \oplus \Gamma(W_0)$ by

$$u(a, \phi) = (a - 2u^{-1}du, u\phi).$$

- Fix $k \geq 4$.
  $\mathcal{G} \leftarrow L^2_{k+2}$-completion
  $\mathcal{C} \leftarrow L^2_{k+1}$-completion
- Chern-Simons-Dirac functional, $CSD \colon \mathcal{C} \to \mathbb{R}$,

$$CSD(a, \phi) = \frac{1}{2}\left(-\int_Y a \wedge da + \int_Y \langle \phi, \partial_a \phi \rangle d\,\mathrm{vol}\right).$$

- If $b_1 = 0 \Rightarrow$ CSD is $\mathcal{G}$-invariant.

$$CSD(u(a, \phi)) = CSD(a, \phi).$$

SWF homology $=$ the Morse homology of CSD

The SWF homotopy type is defined as a Conley index for CSD via finite dimensional approximations:

$$CSD \longrightarrow \text{(finite dim. approx)} \longrightarrow \text{Conley index } \mathrm{SWF}(Y, c).$$

Then,

$$\begin{aligned} \tilde{H}_*(\mathrm{SWF}(Y,c)) &\cong \text{the Morse homology of } CSD \\ &\cong \text{the SWF homology for } (Y,c). \end{aligned}$$

The gradient vector field of CSD w.r.t. L^2-metric

$$\nabla CSD(a, \phi) = (*da + \tau(\phi, \phi), \partial_a \phi).$$

$\nabla CSD(a, \phi) = 0 \Leftrightarrow$ 3-dim. Seiberg-Witten eqns on (Y, c)

$$\mathrm{Crit}(CSD) = \{\text{solutions to SW}\}.$$

Seiberg-Witten trajectories

= trajectories of the downward grad. flow of *CSD*.

$$x = (a, \phi) \colon \mathbb{R} \to \mathcal{C},$$

$$\frac{\partial}{\partial t} x(t) = -\nabla CSD(x(t)) \quad \cdots (\bigstar)$$

$(\bigstar) \Leftrightarrow$ 4-dim. Seiberg-Witten eqns on $Y \times \mathbb{R}$

Definition

A SW-trajectory $x(t)$ is of finite type

$\overset{\text{def}}{\Leftrightarrow}$ $CSD(x(t))$ & $\|\phi_t\|_{C^0}$ are bounded functions in t.

Proposition (Compactness)

$\forall m \in \mathbb{Z}_{>0}$ $\exists C_m$ $\forall$ finite type traj. $x(t) = (a_t, \phi_t)$ s.t.

$$\forall t \ \exists u_t \in \mathcal{G}, \ \|u_t(a_t, \phi_t)\|_{C^m} \le C_m.$$

Projection to the Coulomb gauge

- CSD is $\mathcal{G}$-invariant $\Rightarrow$ Want to consider $CSD_{/\mathcal{G}} \colon \mathcal{C}/\mathcal{G} \to \mathbb{R}$.
- Instead of dividing by $\mathcal{G}$, project to the slice at $(0,0)$.

$$\mathcal{G}_0 := \left\{ u = e^{i\xi} \,\middle|\, \xi \colon Y \to \mathbb{R}, \int_Y \xi = 0 \right\},$$

$$\mathcal{G}_0 \curvearrowright \mathcal{C} \text{ free},$$

$$\mathcal{G}/\mathcal{G}_0 = S^1 \leftarrow \text{the stabilizer of } (0,0).$$

- $V := i \ker d^* \oplus \Gamma(W_0)$. $\leftarrow$ The slice at $(0,0)$

$$\Rightarrow \forall (a,\phi) \in \mathcal{C}, \exists_1 u \in \mathcal{G}_0, u(a,\phi) \in V.$$

- This gives the Coulomb projection $\Pi \colon \mathcal{C} \to V$.

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└Seiberg-Witten trajectories

- Choose a metric $\tilde{g}$ on V s.t.

$$\Pi' \circ \nabla^{L^2}(CSD) = \nabla^{\tilde{g}}(CSD|_V),$$

where Π' = the differential of Π. Then,

$$\boxed{\Pi\text{-projection of } \nabla^{L^2}(CSD) \text{ trajectory} \leftrightarrow \text{Traj. of } \nabla^{\tilde{g}}(CSD|_V).}$$

- Note $\nabla^{\tilde{g}}(CSD|_V)$ is S^1-equivariant.

- Decompose $\nabla^{\tilde{g}}(CSD|_V)$ as $\nabla^{\tilde{g}}(CSD|_V) = l + c$, where
  - l: linear, $l(a,\phi) = (*da, \partial_0\phi)$,
  - c: quadratic, compact.
- We concentrate on trajectories

$$\boxed{x \colon \mathbb{R} \to V, \ \frac{\partial}{\partial t}x(t) = -(l+c)(x(t)).}$$

Finite dimensional approximation

- $l(a, \phi) = (*da, \partial_0 \phi)$: self-adjoint $\Rightarrow$ has real eigenvalues.

$$V_\lambda^\mu := \bigoplus_{\nu \in (\lambda, \mu]} \ker(l - \nu \, \mathrm{id}_V),$$

$$\tilde{p}_\lambda^\mu \colon V \to V_\lambda^\mu (\subset V). \leftarrow L^2\text{-projection}$$

- When varying λ & μ, $\tilde{p}_\lambda^\mu$ may jump.
  $\Rightarrow$ smoothing

$$p_\lambda^\mu \colon V \to V.$$

- Finite dimensional approximation of SW-trajectory is given by,

$$x \colon \mathbb{R} \to V_\lambda^\mu,$$

$$\frac{\partial}{\partial t} x(t) = -(l + p_\lambda^\mu c) x(t).$$

- By Compactness Proposition,

$$\exists R \gg 1, \text{ s.t. } (\forall \text{ finite type traj. of } l + c) \subset B(R),$$

where $B(R)$ is the open ball in $L^2_{k+1}(V)$ with radius R centered at 0.
Fix such an R.

Proposition

For sufficient large $-\lambda$ *&* μ*,*
$x(t)$*: a trajectory of* $l + p^{\mu}_{\lambda} c$*,*

$$\forall t, x(t) \in \overline{B(2R)} \Rightarrow \forall t, x(t) \in B(R).$$

Next, Gradient flows of $l + p^{\mu}_{\lambda} c \Rightarrow$ Conley index.

The Conley index

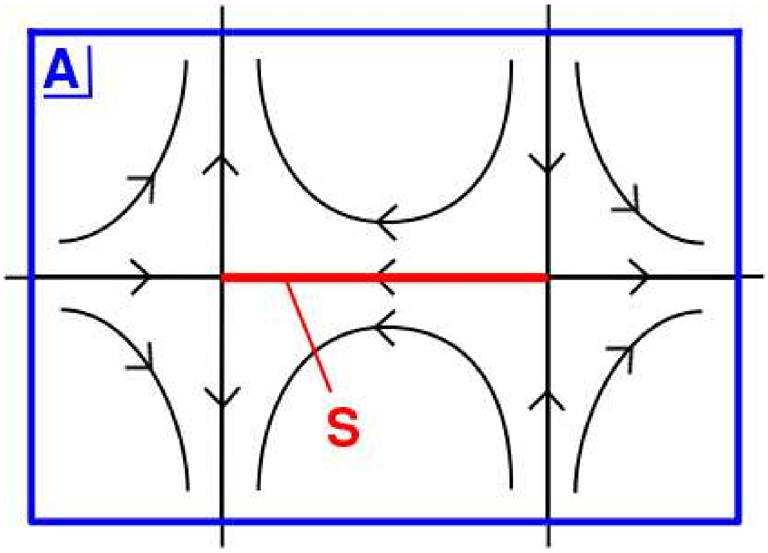
- M: finite dim. manifold
- $\varphi\colon M \times \mathbb{R} \to M$, a continuous flow, $\begin{cases} \varphi_0 = \mathrm{id}, \\ \varphi_{s+t} = \varphi_s \circ \varphi_t. \end{cases}$
- $S(\subset M)$ is invariant $\overset{def}{\Leftrightarrow} \forall t\ \varphi_t(S) = S$.
- For $A(\subset M)$, the invariant set of A is,

$$\mathrm{Inv}(A) := \bigcap_t \varphi_t(A) = \{x \in A \,|\, \forall t\ \varphi_t(x) \in A\}.$$

- A compact subset $S(\subset M)$ is an isolated invariant set if

$$\exists \text{ cpt set } A \text{ s.t. } S = \mathrm{Inv}(A) \subset \mathrm{Int}(A).$$

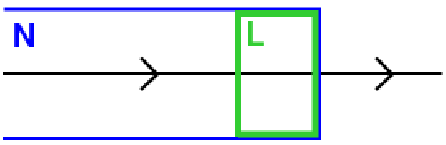
A is called an isolating neighborhood.



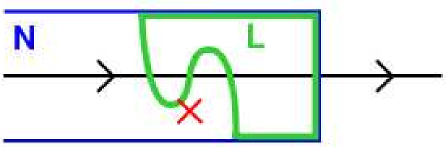
Definition

A pair of compact sets (N, L), $L \subset N \subset M$, is an index pair for an inv. set S if

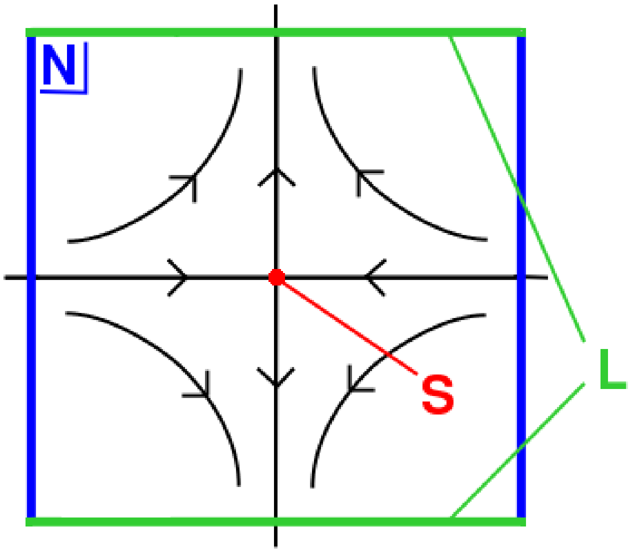
1. $S = \mathrm{Inv}(N \setminus L) \subset \mathrm{Int}(N \setminus L)$.
2. L is an exit set:
   $\forall x \in N, \forall t > 0$ s.t. $\varphi_t(x) \notin N \Rightarrow \exists \tau \in [0, t)\ \varphi_\tau(x) \in L$.



3. L is positively invariant:
   $x \in L,\ t > 0,\ \varphi_{[0,t]}(x) \subset N \Rightarrow \varphi_{[0,t]}(x) \subset L$.



An example of index pair



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└The Conley index

Theorem (Conley)

For every iso. inv. set S & isolating nbd. A,

$$\exists \text{ index pair } (N, L) \text{ for } S \text{ s.t. } N \subset A.$$

Definition

The Conley index of S is,

$$I(\varphi, S) := \text{ the pointed homotopy type of } (N/L, [L]).$$

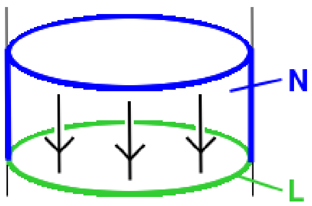
Properties

- $I(\varphi, S)$ is independent of the choice of (N, L).
- (Continuation)
  φ^λ $(\lambda \in [0,1])$: continuous family of flows.
  S^λ: invariant sets for φ_λ.
  If $\forall\lambda$ A is an isolating nbd. of S_λ, i.e., $\mathrm{Inv}(A) = S^\lambda$,

$$\Rightarrow \quad I(\varphi^0, S^0) \cong I(\varphi^1, S^1).$$

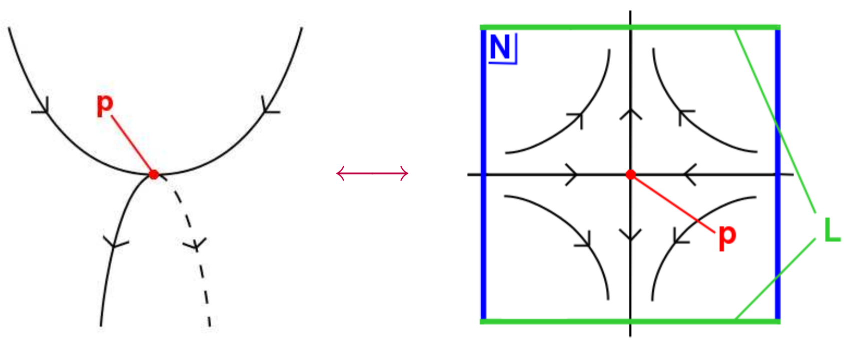
Examples

1. $I(\varphi, \emptyset) = \{1\mathrm{pt}\}$



2. If p is a crit. point of a grad. flow with index$=k$,

$$I(\varphi, \{p\}) \cong S^k.$$



3. M: closed,
   $f: M \to \mathbb{R}$, Morse function s.t. Morse-Smale,
   $S := \{$Crit. points of f & trajectories between them$\}$

$$\Rightarrow \quad \tilde{H}_*(I(\varphi, S)) \cong \text{ the Morse homology of } f.$$

Theorem (Floer et al.)

- *G: a compact Lie group,*
- *$G \curvearrowright M$ preserving φ_t,*
- *S: a G-invariant iso. inv. set.*

Then, the G-equivariant Conley index $I_G(\varphi, S)$ is defined as a pointed G-homotopy type.

Construction of the invariant

Finite dim. approx. $x\colon \mathbb{R} \to V_\lambda^\mu$ is "stable" when $-\lambda$ & $\mu \to \infty$.

$\Rightarrow$ The Conley index is also stable.

$\Rightarrow$ SWF(Y, c) is defined as an object in a certain stable homotopy category.

Let S^1 act on $\mathbb{R}$ & $\mathbb{C}$ as:

- $S^1 \curvearrowright \mathbb{R}$: trivially
- $S^1 \curvearrowright \mathbb{C}$: by multiplication.

Definition

$\mathcal{C}$ is S^1-equivariant graded suspension category as follows:

- Object: (X, m, n)
  X: pointed S^1-space, $m \in \mathbb{Z}$, $n \in \mathbb{Q}$.
- Morphism:

$$\{(X, m, n), (X', m', n')\}_{S^1} = \begin{cases} \emptyset & \text{if } n - n' \notin \mathbb{Z}, \\ \operatorname*{colim}_{k,l} \left[(\mathbb{R}^k \oplus \mathbb{C}^l)^+ \wedge X, (\mathbb{R}^{k+m-m'} \oplus \mathbb{C}^{l+n-n'})^+ \wedge X' \right]_{S^1}, & \text{if } n - n' \in \mathbb{Z}. \end{cases}$$

Note: $(X, m, n) \cong (\mathbb{R}^+ \wedge X, m+1, n) \cong (\mathbb{C}^+ \wedge X, m, n+1)$.

$$(X, m, n) \mapsto (X, m-1, n) \cong (\mathbb{R}^+ \wedge X, m, n)$$
$$(X, m, n) \mapsto (X, m, n-1) \cong (\mathbb{C}^+ \wedge X, m, n)$$

- $m \mapsto m+1 \leftrightarrow$ (formal) desuspension by $\mathbb{R}^+$
- $n \mapsto n+1 \leftrightarrow$ (formal) desuspension by $\mathbb{C}^+$
- Denote $(X, 0, 0)$ by X.
- For a finite dim. vector space E with trivial S^1-action,

$$\Sigma^{-E} X := (E^+ \wedge X, 2\dim_{\mathbb{R}} E, 0).$$

- For a finite dim. vector space E with free S^1-action except 0,

$$\Sigma^{-E} X := (X, 0, \dim_{\mathbb{C}} E).$$

- Recall our situation.

$$\left.\begin{aligned} x\colon \mathbb{R} \to V_\lambda^\mu, \\ \frac{\partial}{\partial t} x(t) = -(l + p_\lambda^\mu c) x(t), \end{aligned}\right\} \longrightarrow \text{gradient flow } \varphi_\mu^\lambda.$$

- Define the isolated invariant set S_λ^μ by

$$\begin{aligned} S_\lambda^\mu :=& \operatorname{Inv}(V_\lambda^\mu \cap \overline{B(2R)}) \\ =& \{\text{Crit. points in } B(R) \text{ \& trajectories connecting them}\} \end{aligned}$$

$$\longrightarrow \boxed{I_\lambda^\mu = I_{S^1}(\varphi_\mu^\lambda, S_\lambda^\mu).}$$

Definition

$$\boxed{\mathrm{SWF}(Y,c) := \left(\Sigma^{-V_\lambda^0} I_\lambda^\mu,\ 0,\ n(Y,c,g)\right),}$$

where $n(Y,c,g)$ is a rational number determined from eta invariants of Dirac & sign.

Why $\Sigma^{-V_\lambda^0} I_\lambda^\mu$ & what is $n(Y,c,g)$?

$$\begin{aligned}\text{Morse index of the reducible } (0,0) &= \#\{\text{negative eigenvalues}\}\\ &= \dim V_\lambda^0. \leftarrow \text{ depending on } \lambda,\ g\end{aligned}$$

$$\Rightarrow \boxed{\Sigma^{-V_\lambda^0} I_\lambda^\mu}$$

- g_t: path of metric $g_0 \overset{g_t}{\sim} g_1$.
- If λ is not eigenvalue for $\forall g_t$,

$$\begin{aligned}\Rightarrow \dim(V_\lambda^0)_{g_1} - \dim(V_\lambda^0)_{g_0} =& SF((\partial)_{g_t})\\ =& n(Y,c,g_1) - n(Y,c,g_0).\end{aligned}$$

$$\Rightarrow \boxed{\Sigma^{-V_\lambda^0 - \mathbb{C}^{n(Y,c,g)}} I_\lambda^\mu}$$

Theorem

$\mathrm{SWF}(Y, c) = \Sigma^{-V^0_\lambda - \mathbb{C}^{n(Y,c,g)}} I^\mu_\lambda$ *is independent of parameters.*

Example

If Y admits a metric of positive scaler curvature

$\Rightarrow$ The reducible θ is the unique solution.

$\Rightarrow S^\mu_\lambda = \{\theta\}$. $\quad \Rightarrow I^\mu_\lambda = (V^0_\lambda)^+$.

$$\Rightarrow \boxed{\mathrm{SWF}(Y, c) = \left(\mathbb{C}^{-n(Y,c,g)}\right)^+.}$$

- $Y = S^3 \Rightarrow \mathrm{SWF}(Y) = S^0$.
- $Y =$ Poincaré sphere $\Rightarrow \mathrm{SWF}(Y) = \mathbb{C}^+$.

Relative Bauer-Furuta invariants

First, we recall ordinary Bauer-Furuta invariants.

Bauer-Furuta invariants

Bauer-Furuta invariant is a stable cohomotopy refinement of the Seiberg-Witten invariant defined by [Bauer-Furuta].

- X: closed ori. 4-mfd. For simplicity, suppose $b_1 = 0$.
- $\hat{c}$: Spin^c-structure on X.
- Fix a connection $\hat{A}_0$ on the determinant line bundle $\det \hat{c}$ of $\hat{c}$.

- Monopole map

$$SW\colon i\ker d^*(\subset i\Omega^1(X)) \oplus \Gamma(W^+) \to i\Omega^+(X) \oplus \Gamma(W^-)$$
$$(\hat{a}, \hat{\phi}) \mapsto (F^+_{\hat{A}_0+\hat{a}} + \sigma(\hat{\phi}, \hat{\phi}^*), D_{\hat{A}_0+\hat{a}}\hat{\phi})$$

- Decompose SW as

$$SW = L + C,$$

where L: linear & C: quadratic, compact.

- Take a finite dim. approx. of $L + C$:

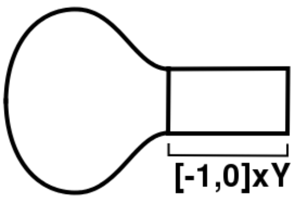
$$L + p^\nu C\colon U' \to U.$$

- The Bauer-Furuta invariant $\mathrm{BF}_X(c)$ is defined as

$$\mathrm{BF}_X(c) = [L + p^\nu C] \in \left\{ \left(\mathbb{C}^{\mathrm{ind}_\mathbb{C} D_{\hat{A}_0}}\right)^+, \left(\mathbb{R}^{b^+}\right)^+ \right\}_{S^1}.$$

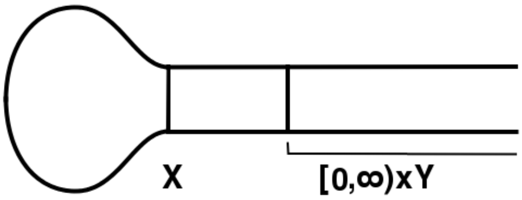
Relative Bauer-Furuta invariants

- Y: closed ori. 3-manifold, $b_1(Y) = 0$.
- X: compact ori. 4-manifold, $\partial X = Y$.
  For simplicity, $b_1(X) = 0$.
- Fix a metric $\hat{g}$ as in the picture below.
- $\hat{c}$: Spin^c-structure on X. $\to c := \hat{c}|_Y$, a Spin^c-str. on Y.
- Fix a connection $\hat{A}_0$ on $\det\hat{c}$ s.t. $A_0 := \hat{A}_0|_Y$ is a flat conn. on $\det c$.

$X =$ 

(Ordinary) relative SW-invariants (a rough sketch)

- Let $\hat{X} = X \cup_Y [0, \infty) \times Y$:

$\hat{X} =$ 

- Consider L^2-moduli $\mathcal{M}^{L^2}(\hat{X}, \hat{c})$.

$$\partial_\infty \colon \mathcal{M}^{L^2}(\hat{X}, \hat{c}) \to \mathcal{M}(Y, c) = \mathrm{Crit}(CSD).$$

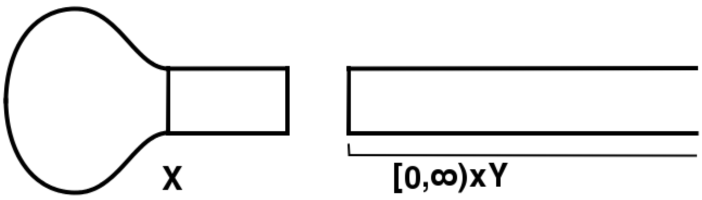
- Note $HF_*^{SW}(Y, c)$ is generated by crit pts of CSD.
- Relative SW-invariant $\Psi_{X,\hat{c}} \in HF_*^{SW}(Y, c)$,

$$\Psi_{X,\hat{c}} := \sum_{a \in \mathrm{Crit}(CSD)} \#(\partial_\infty^{-1}(a))\langle a \rangle \in HF_*^{SW}(Y, c)$$

Seiberg-Witten Floer homotopy type
└Relative Bauer-Furuta invariants

The Idea for relative BF invariants

- Decompose $\hat{X}$ into X and the cylinder of Y:



- Then,

$$\text{A SW-solution } (\hat{A}, \hat{\phi}) \text{ on } \hat{X}$$
$$= (\hat{A}, \hat{\phi})|_X + \Big(\text{a flow } x(t) \text{ on } V \text{ with } x(0) = (\hat{A}, \hat{\phi})|_{\{0\} \times Y} \Big).$$

Monopole map

- $\Omega^1_{\hat{g}}(X) := \{\hat{a} \in \Omega^1(X) \mid \hat{a} \in \ker d^*, \hat{a}|_{\partial X}(\nu) = 0\}$,
  where ν: the unit normal vector to the boundary Y.
- Fix a large μ.
- Monopole map

$$SW\colon i\Omega^1_{\hat{g}}(X) \oplus \Gamma(W^+) \to i\Omega^+(X) \oplus \Gamma(W^-) \oplus V^{\mu}_{-\infty},$$
$$SW(\hat{a}, \hat{\phi}) = (F_{\hat{A}_0+\hat{a}} + \sigma(\hat{\phi}, \hat{\phi}), D_{\hat{A}_0+\hat{a}}\hat{\phi}, p^{\mu}\Pi i^*(\hat{a}, \hat{\phi})),$$

where i^* : the restriction to $\partial X = Y$,
Π : the Coulomb projection to V,
$p^{\mu}\colon V \to V^{\mu}_{-\infty}$, the L^2-projection.

- Write briefly as $SW\colon \mathcal{C}_X \to \mathcal{U} \oplus V^{\mu}_{-\infty}$.

- Decompose $SW\colon \mathcal{C}_X \to \mathcal{U} \oplus V^{\mu}_{-\infty}$ as $SW = L + C$,
  where L: linear & C: quadratic.
- Take a finite dimensional subspace $U \subset \mathcal{U}$, and fix $\lambda \ll 0$.
  $\Rightarrow$ Put $U' := L^{-1}(U \times V^{\mu}_{\lambda})$, and

$$SW_U := L + \mathrm{pr}_{U \times V^{\mu}_{\lambda}} C\colon U' \to U \times V^{\mu}_{\lambda}.$$

- Fix small $\varepsilon > 0$. Let $B(U, \varepsilon) \subset U$ be the ε-ball in U.

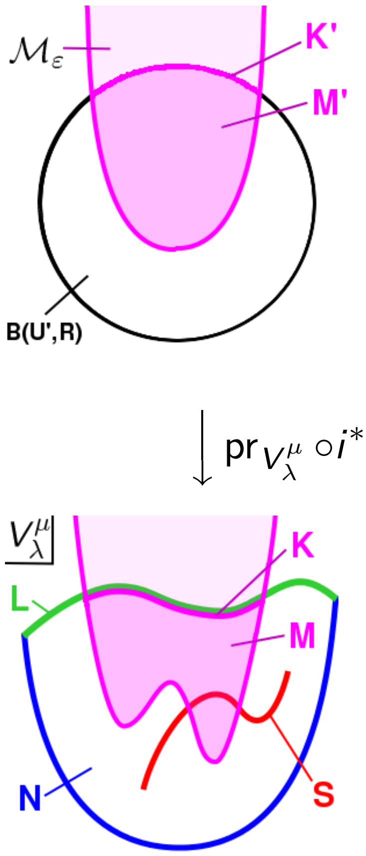
$$\mathcal{M}_{\varepsilon} := SW_U^{-1}(B(U, \varepsilon) \times V^{\mu}_{\lambda}).$$ ← Almost the SW-moduli

- Fix $R' \gg 1$.
  Let $B(U', R') \subset U'$ be the R'-ball.
  $S(U', R') = \partial B(U', R')$.

$$M' := \mathcal{M}_\varepsilon \cap B(U', R'),$$
$$K' := \mathcal{M}_\varepsilon \cap S(U', R'),$$

$$M := \mathrm{pr}_{V_\lambda^\mu} \circ i^*(M'),$$
$$K := \mathrm{pr}_{V_\lambda^\mu} \circ i^*(K').$$

$\Rightarrow$ Can find an index pair (N, L) s.t. $M \subset N$, $K \subset L$.



Seiberg-Witten Floer homotopy type
└ Relative Bauer-Furuta invariants

$\Rightarrow SW_U \colon B(U', R')/S(U', R') \to (B(U, \varepsilon)/S(U, \varepsilon)) \wedge N/L.$
$\Rightarrow \Psi_{X,\hat{c}} \colon (U')^+ \to U^+ \wedge I_\lambda^\mu.$

The relative Bauer-Furuta invariant is,

$$BF_X(\hat{c}) = [\Psi_{X,\hat{c}}] \in \left\{ \left(S^k, b^+(X), n(X, c, g) \right), \mathrm{SWF}(Y, c) \right\}_{S^1}$$

where $k = \mathrm{ind}_{APS}\, D_{\hat{A}}$.

Gluing formula for relative BF invariants

S-duality for Conley indices

- $f : M \longrightarrow \mathbb{R}$, Morse function of a n-dim. mfd.
  $\Rightarrow$ Morse flow φ_f.
  $\Rightarrow$ The Conley index $I_f = I(\varphi_f, S)$.
- $-f \Rightarrow$ the reverse flow φ_{-f}.
  $\Rightarrow$ The Conley index $I_{-f} = I(\varphi_{-f}, S)$.

Theorem (Cornea)

I_{-f} is a Spanier-Whitehead n-dual of I_f, i.e.,

$$\exists \eta \colon I_f \wedge I_{-f} \longrightarrow S^n, \ \textit{n-duality map s.t.}$$
$$\eta^* \colon \{S^0, I_f\} \xrightarrow{\cong} \{I_{-f}, S^n\}.$$

Note For $\alpha \in \{S^0, I_f\}$, $\eta^*(\alpha)$ is given as follows:

$$S^0 \wedge I_{-f} \xrightarrow{\alpha \wedge \mathrm{id}} I_f \wedge I_{-f} \xrightarrow{\eta} S^n.$$

Corollary

SWF$(-Y)$ is a S-dual of SWF(Y).

$(\because)$ $CSD_{-Y} = -CSD_Y$.

Gluing formula

- $X = X_1 \cup_Y X_2$.
- η: $\mathrm{SWF}(Y) \wedge \mathrm{SWF}(-Y) \to S^0$, the duality map.

$$\begin{aligned}
\mathrm{BF}_X &= [\Psi_X] \in \left\{(S^0, b^+(X), -d), S^0\right\}_{S^1},\\
\mathrm{BF}_{X_1} &= [\Psi_{X_1}] \in \left\{(S^0, b^+(X_1), -d_1), \mathrm{SWF}(Y)\right\}_{S^1},\\
\mathrm{BF}_{X_2} &= [\Psi_{X_2}] \in \left\{(S^0, b^+(X_2), -d_2), \mathrm{SWF}(-Y)\right\}_{S^1}.
\end{aligned}$$

$$\Rightarrow S^\bullet \wedge S^\bullet \xrightarrow{\Psi_{X_1} \wedge \Psi_{X_2}} \mathrm{SWF}(Y) \wedge \mathrm{SWF}(-Y) \xrightarrow{\eta} S^\bullet.$$

Theorem (Gluing formula)

$$\Psi_X \simeq \eta \circ (\Psi_{X_1} \wedge \Psi_{X_2}).$$

Cobordism

- X: a compact 4-manifold, $\partial X = (-Y_1) \cup Y_2$.

$$\begin{array}{ccc}
\Psi_X & \in & \{(S^0, b, -d), \mathrm{SWF}(-Y_1) \wedge \mathrm{SWF}(Y_2)\}_{S^1} \\
\downarrow & & \downarrow S\text{-duality} \ \cong \\
\mathcal{D}(X) & \in & \{(\mathrm{SWF}(Y_1), b, -d), \mathrm{SWF}(Y_2)\}_{S^1}
\end{array}$$

A cobordism X gives a morphism $\mathcal{D}(X)$ between SWF's.

Theorem

- X_a: $\partial X_a = (-Y_1) \cup Y_2$ & X_b: $\partial X_b = (-Y_2) \cup Y_3$.
- $X = X_a \cup_{Y_2} X_b$.

$$\Rightarrow \mathcal{D}(X) = \Sigma^{b^+(X_a), -d(X_a)} \mathcal{D}(X_b) \circ \mathcal{D}(X_a)$$

Applications

to negative definite manifolds

Theorem A (Donaldson)

X: closed ori. 4-mfd with negative definite form Ψ_X
$\Rightarrow \Psi_X \cong$ diagonal.

Proof by [Bauer-Furuta]

The finite dim approx. of the monopole map:

$$f\colon S^V := (\mathbb{R}^m \oplus \mathbb{C}^{n+s})^+ \to (\mathbb{R}^m \oplus \mathbb{C}^n)^+.$$

This is S^1-equivariant and $\deg f|_{(S^V)^{S^1}} = 1$.

Lemma

For f as above, $s \leq 0$.

($\because$) Use tom Dieck's character formula.

Proof of Theorem A.

$$0 \geq s = \operatorname{ind}_{\mathbb{C}} D = \frac{c^2 + b_2(X)}{8}.$$

$\therefore \forall c$: characteristic $c^2 + b_2(X) \leq 0$.
$\underset{\text{[Elkies]}}{\Rightarrow}$ Diagonal. □

Question

Does an analogue of Theorem A for 4-mfd with boundary?

→ Yes.

Theorem (Froyshov)

X: compact ori. 4-mfd, ∂X = the Poincaré 3-sphere.
If $\Psi_X \cong m(-1) \oplus J$, where J:negative definite even,

$$\Rightarrow J = 0 \text{ or } J = -E_8.$$

- Froyshov proved this by the invariant he defined.
- This can be proved by using SWF.

Seiberg-Witten Floer homotopy type
└ Applications

- X: negative definite, $\partial X = Y$. The monopole map f satisfies

$$[f] \in \{(\mathbb{C}^s)^+, \mathrm{SWF}(Y,c)\}_{S^1} \ \& \ \deg f^{S^1} = 1.$$

$$s(Y,c) := \max\left(s \,\middle|\, \exists f \text{ s.t.} \begin{Bmatrix} [f] \in \{(\mathbb{C}^s)^+, \mathrm{SWF}(Y,c)\}_{S^1} \\ deg f^{S^1} = 1 \end{Bmatrix} \right).$$

- Recall, if Y is the Poincare 3-sphere $\Rightarrow \mathrm{SWF}(Y) = \mathbb{C}^+$.
  Then, $\boxed{s(Y) = 1}$.
  $\because f \in \{(\mathbb{C}^s)^+, \mathbb{C}^+\}_{S^1}$ & $\deg f^{S^1} = 1 \Rightarrow s - 1 \leq 0$.

Theorem

X: compact ori. negative definite, $\partial X = Y$.
For $\forall$ characteristic c

$$\frac{b_2(X) + c^2}{8} \le s(Y, c),$$

Proof.

The monopole map f for c satisfies

$$[f] \in \{(\mathbb{C}^k)^+, \mathrm{SWF}(Y, c)\}_{S^1} \ \& \ \deg f^{S^1} = 1,$$

where $k = (c^2 + b_2)/8$.
Therefore $k \le s(Y, c)$. □

Theorem (Froyshov)

X: compact ori. 4-mfd, ∂X = the Poincaré 3-sphere.
If $\Psi_X \cong m(-1) \oplus J$, where J:negative definite even,

$$\Rightarrow J = 0 \text{ or } J = -E_8.$$

Proof

- Note that $c = (\overbrace{1, \dots, 1}^{m}, 0, \dots, 0)$ is a characteristic.
  $\Rightarrow \operatorname{rank} J = b_2 - m = b_2 + c^2 \le 8s(Y) = 8$.
  $\therefore J = 0$ or $J = -E_8$. □

Question

What is the relation between $s(Y)$ & Froyshov's invariants?