Families Seiberg-Witten invariants and topology of spin families of 4-manifolds

joint work with T. Kato and H. Konno

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Main Theorem (Kato-Konno-N., 2019.6)

M: closed smooth spin 4-manifold

$$M \underset{\text{homeo.}}{\cong} K3\#nS^2 \times S^2 \ (0 \leq n \leq 3)$$

$\Rightarrow \mathrm{Diff}(M) \overset{\iota}{\hookrightarrow} \mathrm{Homeo}(M)$ is not a weak homotopy equivalence.

$\exists i \leq n \quad \pi_i\mathrm{Diff}(M) \xrightarrow{\iota_*} \pi_i\mathrm{Homeo}(M)$ is not an isomorphism.

Remark

- $\mathrm{Diff}(M) \leftarrow C^\infty$-topology
- $\mathrm{Homeo}(M) \leftarrow C^0$-topology
- If $\dim M \leq 3$,
  $\Rightarrow \mathrm{Diff}(M) \hookrightarrow \mathrm{Homeo}(M)$ is a weak homotopy equivalence.
  $\dim M \leq 2 \rightarrow$ Classical
  $\dim M = 3 \rightarrow$ [Hatcher '83]
- $n = 0$: Essentially known $\leftarrow$ a collorary of the Well-known fact below

Theorem (Baraglia, 2019.7)

M: closed smooth 4-manifold, $\pi_1 M = 1$, $|\operatorname{sign}(M)| > 8$

$$b := \begin{cases} \min\{b_+, b_-\} - 1 & \text{if } M\text{: nonspin} \\ \min\{b_+, b_-\} - 3 & \text{if } M\text{: spin} \end{cases}$$

$\Rightarrow \quad \exists i \leq b \quad \pi_i \mathrm{Diff}(M) \xrightarrow{\iota_*} \pi_i \mathrm{Homeo}(M)$ is not an isomorphism.

Idea of the proof

Construct a nonsmoothable fiber bundle
$$\begin{array}{ccc} M & \to & X \\ & & \downarrow \\ & & T^{n+1} \end{array}$$

- the structure group is in $\mathrm{Homeo}(M)$, & cannot be reduced to $\mathrm{Diff}(M)$

$$\begin{array}{ccc} & & B\,\mathrm{Diff}(M) \\ & \overset{\nexists}{\nearrow} & \downarrow \\ T^{n+1} & \xrightarrow[\phi]{} & B\,\mathrm{Homeo}(M) \end{array}$$

where ϕ is the classifying map of $X \to T^{n+1}$.

$\Rightarrow \exists i \leq n \quad \pi_i(\mathrm{Homeo}(M)/\,\mathrm{Diff}(M))$ nontrivial

To prove nonsmoothability,

- [Kato-Konno-N.] a family version of $10/8$-type inequality
- [Baraglia] a family version of "Diagonalization theorem"

Well-known fact([Donaldson '90]...)

A homotopy $K3$ surface K admits NO self-diffeo. s.t.

$$(*)\begin{cases}\text{preserving the ori. of } K\\ \text{reversing the ori. of } H^+(K)\end{cases}$$

Remark 1 The above fact $\Rightarrow \pi_0 \operatorname{Diff}(K) \neq \pi_0 \operatorname{Homeo}(K)$

Remark 2 $K\#(S^2 \times S^2)$ admits a self-diffeo. s.t. $(*)$

$(\because)$ $S^2 \times S^2$ admits a self-diffeo ρ s.t. $(*)$.

$\mathrm{id}_K \#\rho$ satisfies $(*)$.

Corollary

M: spin 4-manifold with $\operatorname{sign}(M) = -16$ & $\pi_1 M = 1$

If M admits a self-diffeo. s.t. $(*)$, then

$$b^+(M) \geq 4$$

- $M = K3\#n(S^2 \times S^2) \Rightarrow b_+(M) = 3 + n$
- Define commuting self-diffeos. $f_1, \dots, f_n$ of M by

$$f_i = \mathrm{id}_{K3} \# \mathrm{id}_{S^2\times S^2} \# \cdots \# \underset{\substack{\uparrow \\ i\text{th } S^2\times S^2}}{\rho} \# \cdots \# \mathrm{id}_{S^2\times S^2}$$

$\longrightarrow$ A multiple mapping torus $\begin{array}{l} X = (M \times [0,1]^n)/f_1, \cdots, f_n \\ \downarrow \\ T^n \end{array}$

- $\exists$ spin str. on the tangent bundle along fiber, $T(X/T^n)$.
  $\rightarrow$ a family of Dirac operators
- $H^+ \to T^n$: the bundle of $H^+(M)$.

Proposition $\mathrm{ind}\, D = [\underline{\mathbb{H}}]$, $H^+ = \underline{\mathbb{R}}^3 \oplus \xi_n$
$\xi_n = \pi_1^*\ell \oplus \cdots \oplus \pi_n^*\ell$, $\pi_i : T^n = S^1 \times \cdots \times S^1 \to S^1$ ith proj.
$\ell \to S^1$, nontrivial $\mathbb{R}$-bundle

A $\frac{10}{8}$-type inequality

Theorem B (Kato-Konno-N.)

Suppose

- M: closed spin 4-manifold with $\mathrm{sign}(M) = -16$, $b_1(M) = 0$
- $\begin{array}{c} M \to X \\ \downarrow \\ T^{n+1} \end{array}$: fiber bundle with structure group $\mathrm{Diff}(M)$
- $\exists$ spin str. on $T(X/T^n)$.
- $[\mathrm{ind}\, D] = [\underline{\mathbb{H}}]$, $[H^+] = [\underline{\mathbb{R}}^a \oplus \xi_n]$ in $KO_{\mathrm{Pin}(2)}(T^n)$
  $\Rightarrow b_+(M) = a + n$

Then

$$b_+(M) \geq 3 + n \ (a \geq 3)$$

- [Freedman] $K3 \underset{\text{homeo}}{\cong} 2|E_8|\#3(S^2 \times S^2)$
- Let $M \underset{\text{homeo}}{\cong} 2|E_8|\#a(S^2 \times S^2)\#m(S^2 \times S^2) \quad (m \geq 1)$
- Define commuting self-homeos. $f_1, \ldots, f_m$ of M by

$$f_i = \mathrm{id}_{2|E_8|\#a(S^2\times S^2)} \# \mathrm{id}_{S^2\times S^2} \# \cdots \# \underset{\substack{\uparrow \\ i\text{th } S^2 \times S^2}}{\rho} \# \cdots \# \mathrm{id}_{S^2\times S^2}$$

$\longrightarrow$ A multiple mapping torus $\begin{array}{c} X = (M \times [0,1]^m)/f_1, \cdots, f_m \\ \downarrow \\ T^m \end{array}$

Theorem C (Kato-Konno-N.)

(1) The total space X is smoothable

(2) If $a \leq 2$ & $m \leq 4 \Rightarrow$ the structure group in $\mathrm{Homeo}(M)$ cannot be reduced to $\mathrm{Diff}(M)$.

Proof of Theorem C(1)

- Kirby-Siebenmann theory
- $S^1 \times 2|E_8|$ is smoothable

Theorem B $\Rightarrow$ Theorem C(2)

- Only thing we need to check is $\operatorname{ind} D = [\mathbb{H}]$
- $m \leq 3 \Rightarrow \widetilde{KSp}(T^m) = 0 \Rightarrow$ OK
- $m = 4 \Rightarrow$ use Novikov's theorem (topological invariance of rational Pontrjagin classes).

Other application

Nonsmoothable $\mathbb{Z}^m$-action

$$\begin{array}{ccc} & & B\,\mathrm{Diff}(M) \\ & \overset{\nexists}{\nearrow} & \downarrow \\ T^m = {\color{red}B\mathbb{Z}^m} & \longrightarrow & B\,\mathrm{Homeo}(M) \end{array}$$

$$\begin{array}{ccc} & & \mathrm{Diff}(M) \\ & \overset{\nexists}{\nearrow} & \downarrow \\ \mathbb{Z}^m & \longrightarrow & \mathrm{Homeo}(M) \end{array}$$

$\longrightarrow$ Nosmoothable $\mathbb{Z}^m$-action

- Let $M = 2|E_8|\#(n+3)(S^2 \times S^2)$, $n \geq 1$
- Define commuting self-homeos. $f_1, \ldots, f_{n+3}$ of M by

$$f_i = \mathrm{id}_{2|E_8|} \# \mathrm{id}_{S^2 \times S^2} \# \cdots \# \underset{\substack{\uparrow \\ i\text{th } S^2 \times S^2}}{\rho} \# \cdots \# \mathrm{id}_{S^2 \times S^2}$$

Theorem D (Kato-Konno-N.)

For an arbitrary subset of k homeos

$$\{f_{i_1}, \ldots, f_{i_k}\} \subset \{f_1, \ldots, f_{n+3}\},$$

- $k \leq n \Rightarrow \exists$smooth structure on M s.t. all f_{i_j} smooth
- $k > n \Rightarrow$ No smooth structure on M s.t. all f_{i_j} smooth

Remark

Similar examples

- [N.'09] Nonsmoothable $\mathbb{Z} \times \mathbb{Z}$-action on Enriques$\#S^2 \times S^2$
- [Y. Kato '16] Nonsmoothable $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on some spin manifold
- [Baraglia '18] Various nonsmoothable actions

Outline of the proof of Theorem B

- For $\begin{array}{c} M \to X \\ \quad\ \downarrow \\ \quad\ B \end{array}$ with spin structure on $T(X/B)$, we have the family of the monopole maps over X. Suppose $b_1(M) = 0$.
- By finite dimensional approximation, we have a $\mathrm{Pin}(2)$-equivariant, fiber preserving, proper map f:

$$\begin{array}{ccccc} V & \xrightarrow{f} & \underline{W} & = & W \times B \\ \downarrow & & \downarrow & & \\ B & = & B & & \end{array}$$

where V, $\underline{W}$ vector bundles s.t.

$$[V] - [\underline{W}] = [\mathrm{ind}\, D] - [H^+] \in KO_{\mathrm{Pin}(2)}(B)$$

$f^{-1}(0)/\,\mathrm{U}(1) \doteqdot$ the Seiberg-Witten moduli space

- Suppose $-\frac{\operatorname{sign}(M)}{4} - (1 + b_+(M)) + \dim B = 0$
  $\Rightarrow \dim V = \dim W + 1$, the virtual dim. of the SW moduli $= 0$

$$\begin{array}{ccccc} \{TV, S^W\}^{\mathrm{Pin}(2)} & \xrightarrow{\phi} & \{TV, S^W\}^{\mathrm{U}(1)} & \xrightarrow{\deg} & \mathbb{Z}/2 \\ [f] & \mapsto & [f] & \mapsto & \#(f^{-1}(pt)/\,\mathrm{U}(1)) \end{array}$$

  where TV: Thom space of V
- $\deg f = \#(f^{-1}(pt)/\,\mathrm{U}(1))$ is the family SW invariant.
- $[f]$ is the stable cohomotopy family SW invariant.

Theorem A (Kato-Konno-N.)

1. $\mathrm{Im}(\deg \circ \phi) = \{0\}$ or $\{1\}$
2. Whether 0 or 1 is determined by

$$[V] - [\underline{W}] = [\operatorname{ind} D] - [H^+] \in KO_{\mathrm{Pin}(2)}(B)$$

[Baraglia-Konno] For the aforesaid mapping torus $M \to X \to T^n$, $M = K3\# nS^2 \times S^2$, $\boxed{\deg f = 1}$.

Corollary

For $\begin{array}{ccc} M & \to & X \\ & & \downarrow \\ & & T^n \end{array}$ with spin str. on $T(X/T^n)$,

$$\left.\begin{array}{c} \mathrm{sign}(M) = -16,\ b_1(M) = 0 \\ [\mathrm{ind}\, D] = [\underline{\mathbb{H}}],\ [H^+] = [\underline{\mathbb{R}}^3 \oplus \xi_n] \end{array}\right\} \Rightarrow \mathrm{Im}(\deg \circ \phi) = \{1\}$$

Recall

Theorem B

$$\left.\begin{array}{c} \mathrm{sign}(M) = -16,\ b_1(M) = 0 \\ [\mathrm{ind}\, D] = [\underline{\mathbb{H}}],\ [H^+] = [\underline{\mathbb{R}}^a \oplus \xi_n] \end{array}\right\} \Rightarrow a \geq 3$$

Outline of the proof of Theorem B

- Suppose $a < 3$. May assume $W = W_0 \oplus \mathbb{R}^a$.
- Consider the embedding $W_0 \oplus \mathbb{R}^a \subset W_0 \oplus \mathbb{R}^3$
- $\mathrm{Pin}(2) \to \mathrm{Pin}(2)/\mathrm{U}(1) = \{\pm 1\} \curvearrowright \mathbb{R}^x$. $\mathrm{U}(1) \curvearrowright \mathbb{R}^x$ trivially

$$\begin{array}{ccccc} & & \{TV, S^{W_0\oplus\mathbb{R}^3}\}^{\mathrm{Pin}(2)} & & \\ & \nearrow^{\iota_0} & & \searrow^{\phi} & \\ \{TV, S^{W_0\oplus\mathbb{R}^a}\}^{\mathrm{Pin}(2)} & & & & \{TV, S^{W_0\oplus\mathbb{R}^3}\}^{\mathrm{U}(1)} \xrightarrow{\deg} \mathbb{Z}/2 \\ & \searrow_{\phi} & & \nearrow_{\iota_1} & \\ & & \{TV, S^{W_0\oplus\mathbb{R}^a}\}^{\mathrm{U}(1)} & & \end{array}$$

- By Corollary, $\mathrm{Im}(\deg \circ \phi \circ \iota_0) = \{1\}$
- We can collapse $S^{W_0\oplus\mathbb{R}^a}$ in $S^{W_0\oplus\mathbb{R}^3}$ $\mathrm{U}(1)$-equivariantly.
  $\Rightarrow \iota_1 = 0 \Rightarrow \mathrm{Im}(\deg \circ \iota_1 \circ \phi) = \{0\}$

On Theorem A

$$\begin{array}{ccccc} \{TV, S^W\}^{\mathrm{Pin}(2)} & \xrightarrow{\phi} & \{TV, S^W\}^{\mathrm{U}(1)} & \xrightarrow{\deg} & \mathbb{Z}/2 \\ [f] & \mapsto & [f] & \mapsto & \#(f^{-1}(pt)/\,\mathrm{U}(1)) \end{array}$$

Theorem A (Kato-Konno-N.)

1. $\mathrm{Im}(\deg \circ \phi) = \{0\}$ or $\{1\}$
2. Whether 0 or 1 is determined by

$$[V] - [\underline{W}] = [\mathrm{ind}\, D] - [H^+] \in KO_{\mathrm{Pin}(2)}(B)$$

- $\deg f$ is the family Seiberg-Witten invariant.
- $[f]$ is the stable cohomotopy family SW invariant.

See [Baraglia-Konno] for families SW invariants.

Rigidity of mod 2 SW invariants

[Morgan-Szabó-(Furuta-Kronheimer)'97]

- M: homotopy $K3 \Rightarrow \mathrm{SW}(M, \text{spin}) \underset{(2)}{\equiv} \mathrm{SW}(K3, \text{spin}) = \pm 1$
- M: homotopy $E(2n)$ $(n > 1)$

  $\Rightarrow \mathrm{SW}(M, \text{spin}) \underset{(2)}{\equiv} \mathrm{SW}(E(2n), \text{spin}) = 0$

[Ruberman-Strle'00]

M: homology $T^4 \Rightarrow$ $\mathrm{SW}(M, \text{spin})$ mod 2 is determined by the ring structure of $H^*(M)$

[Bauer'08]

M: almost complex 4-manifold with $c_1(M) = 0$, $b_+(M) \geq 4$
$\Rightarrow \mathrm{SW}(M, \text{spin}) \underset{(2)}{\equiv} 0$

[T.-J.Li'06] Similar results
[Furuta-Kametani-Minami-Matsue'01-07] Stable cohomotopy version

On the proof of Theorem A

$$\{TV, S^W\}^{\mathrm{Pin}(2)} \xrightarrow{\phi} \{TV, S^W\}^{\mathrm{U}(1)} \xrightarrow{\deg} \mathbb{Z}/2$$

- For $\alpha,\ \beta \in \{TV, S^W\}^G$, the equivariant difference obstruction

  $\delta(\alpha, \beta) \in H^k_G(TV, TV^{\mathrm{U}(1)}; \pi_k S^W)$ ← Bredon cohomology

  where $G = \mathrm{Pin}(2)$ or $\mathrm{U}(1)$
- Fix α_0. $k = \dim S^W$. The map $\alpha \mapsto \delta(\alpha, \alpha_0)$ gives a bijection

$$\{TV, S^W\}^G \xrightarrow{1:1} H^k_G(TV, TV^{\mathrm{U}(1)}; \pi_k S^W)$$

- The forgetful map

$$H^k_{\mathrm{Pin}(2)}(TV, TV^{\mathrm{U}(1)}; \pi_k S^W) \to H^k_{\mathrm{U}(1)}(TV, TV^{\mathrm{U}(1)}; \pi_k S^W)$$

  is given by multiplication of 2. $(\mathrm{Pin}(2)/\mathrm{U}(1) = \{\pm 1\})$