Yamabe invariants and $\mathrm{Pin}^{-}(2)$-monopole equations

(Joint work with M. Ishida & S. Matsuo)

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Introduction

- M: closed oriented connected n-manifold $(n \geq 3)$ $\leftarrow$ always assumed
- g: Riemannian metric on M
- $s_g\colon M \to \mathbb{R}$: scalar curvature
- $[g] = \{ug \,|\, u\colon M \to \mathbb{R}^+\}$, conformal class of g
- $\mathcal{C}(M)$: the space of conformal classes

Yamabe invariant

$$\mathcal{Y}(M) = \sup_{[g]\in\mathcal{C}(M)} \inf_{h\in[g]} \frac{\int_M s_h d\mu_h}{V_h^{\frac{n-2}{n}}},$$

where $d\mu_h$ is the volume form of h, and $V_h = \int_M d\mu_h$.

In general, it is difficult to compute the exact values of the Yamabe invariants.

Theorem([LeBrun,'96,'99])

Let M be a compact minimal Kähler surface, $b_+ \geq 2$, $c_1^2(M) \geq 0$. Then

$$\mathcal{Y}(M) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$

- The proof uses the Seiberg-Witten equations.
  A key point is the nontriviality of the SW invariant.

- Note $c_1^2(M) = 2\chi(M) + 3\tau(M)$. χ: Euler, τ: signature

Main theorem

Theorem 1 (Ishida-Matsuo-N.,'14)

Let M be a compact minimal Kähler surface, $b_+ \geq 2$, $c_1^2(M) \geq 0$.
Let $Z = Z_1 \# \cdots \# Z_k$ such that

$$Z_i = S^2 \times \Sigma \quad \text{or} \quad S^1 \times Y^3 \quad \text{or} \quad N$$

with $g(\Sigma) > 0$, $\mathcal{Y}(N) \geq 0$, $b_+(N) = 0$. (Ex. $N = \overline{\mathbb{C}\mathrm{P}^2}$)

Then $$\mathcal{Y}(M\#Z) = \mathcal{Y}(M) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$

- The proof uses the $\mathrm{Pin}^-(2)$-monopole equations.
  (*Cf.* SW equations = $\mathrm{U}(1)$-monopole equations)
- If $\exists Z_i = S^2 \times \Sigma$ or $S^1 \times Y$ s.t $b_1(Y) \geq 1$ ($b_+(Z_i) \geq 1$)
  $\Rightarrow$ SW inv. $\mathrm{SW}^{\mathrm{U}(1)}(M\#Z) \equiv 0$.
  But $\mathrm{Pin}^-(2)$-monopole inv. $\mathrm{SW}^{\mathrm{Pin}^-(2)}(M\#Z) \neq 0$.

Contents

Yamabe invariants

Reference

[K. Akutagawa] *Yamabe invariants*, Sugaku, 66-1(2014), 31–60.

- M: closed oriented connected n-manifold ($n \geq 3$)
- $\mathcal{M}(M) =$ the space of Riemannian metrics on M.

(Normalized) Einstein-Hilbert functional

$$E\colon \mathcal{M}(M) \to \mathbb{R}, \quad g \mapsto \frac{\int_M s_g d\mu_g}{V_g^{\frac{n-2}{n}}}$$

- Critical points of E are Einstein metrics.
- $\inf_g E(g) = -\infty, \quad \sup_g E(g) = +\infty.$

Yamabe constant

- [Fact] For any conformal class $[g]$, $E|_{[g]}$ is bounded below.

Yamabe constant $Y(M,[g]) = \inf_{h\in[g]} E(h)$

Theorem (Yamabe, Trudinger, Aubin, Schoen,...)
$\forall$compact manifold M, $\forall$conformal class $[g]$, $\exists h_0 \in [g]$ s.t.

$$E(h_0) = \inf_{h\in[g]} E(h) = Y(M,[g]).$$

- h is called the Yamabe metric. Then

$$s_h = Y(M,[g]) \cdot V_h^{-\frac{2}{n}} \longleftarrow \text{const.}$$

Yamabe invariant

- [Fact] A critical of E (which is Einstein) is a saddle point.
  $\longrightarrow$ Try min-max!

Definition (O. Kobayashi, Schoen, around '85)

$$\mathcal{Y}(M) = \sup_{[g]\in\mathcal{C}(M)} Y(M,[g]) = \sup_{[g]\in\mathcal{C}(M)} \left(\inf_{h\in[g]} E(h) \right)$$

- If $C \in \mathcal{C}(M)$ attains the sup, and $\mathcal{Y}(M) \leq 0$,
  $\Rightarrow$ the Yamabe metric of C is Einstein.
- But the sup is not necessarily attained in general.

Properties

- Aubin's inequality $\forall M^n$, $\mathcal{Y}(M) \leq \mathcal{Y}(S^n) = \mathcal{Y}(S^n, [h_0])$.
- $\mathcal{Y}(M)$ is a diffeomorphism invariant.
- $n = 3 \Rightarrow \mathcal{Y}(M)$ is a topological invariant.
- $n \geq 4 \Rightarrow \mathcal{Y}(M)$ depends on differential structures.

Ex.
  - $\mathcal{Y}(\mathbb{C}\mathrm{P}^2 \# 9\overline{\mathbb{C}\mathrm{P}}^2) > 0$
  - $\mathcal{Y}(\text{Dolgachev surface}) = 0$
  - $\mathbb{C}\mathrm{P}^2 \# 9\overline{\mathbb{C}\mathrm{P}}^2 \underset{\text{homeo}}{\cong}$ (Dolgachev surface).

- $\mathcal{Y}(M) > 0 \Leftrightarrow M$ admits a positive scalar curvature (PSC) metric.

\- If $\mathcal{Y}(M) > 0 \Rightarrow$ the ordinary & stable cohomotopy SW and $\mathrm{Pin}^{-}(2)$-monopole invariants vanish.

Yamabe invariants of 4-manifolds

- $\mathcal{Y}(S^4) = Y(S^4, [h_0]) = 8\sqrt{6}\pi$
- [LeBrun,'96,'99]
  For a compact minimal Kähler surface M, $b_+ \geq 2$, $c_1^2(M) \geq 0$,
  $$\mathcal{Y}(M) = \mathcal{Y}(M \# k\overline{\mathbb{C}\mathrm{P}}^2) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$
- [LeBrun,'99]
  For a compact Kähler surface M
  $$\begin{cases} \mathcal{Y}(M) > 0 & \Leftrightarrow \quad \mathrm{Kod}(M) = -\infty \\ \mathcal{Y}(M) = 0 & \Leftrightarrow \quad \mathrm{Kod}(M) = 0, 1 \\ \mathcal{Y}(M) < 0 & \Leftrightarrow \quad \mathrm{Kod}(M) = 2 \end{cases}$$

- [Ishida-LeBrun,'03]
  For compact Kähler surfaces M_i $(i = 1, 2, 3, 4)$ s.t.
  $b_1(M_i) = 0$ + some conditions,

$$\mathcal{Y}(\#_{i=1}^k M_i) = \mathcal{Y}(\#_{i=1}^k M_i \# k\overline{\mathbb{C}\mathrm{P}}^2) = -4\sqrt{2}\pi\sqrt{\sum_{i=1}^k c_1^2(M_i)}.$$

  The proof uses the stable cohomotopy SW.
- [Sasahira,'06]
  M_1, M_2, $M_3 = K3$ or $\Sigma_g \times \Sigma_h$ $(g, h$:odd)

$$\mathcal{Y}(\#_{i=1}^k M_i) = \mathcal{Y}(\#_{i=1}^k M_i \# k\overline{\mathbb{C}\mathrm{P}}^2) = -4\sqrt{2}\pi\sqrt{\sum_{i=1}^k c_1^2(M_i)}.$$

  The proof uses the spin bordism SW.

[Sung,'09]

- M: a compact Kähler s.t. $\mathrm{Kod}(M) \geq 0$
  N: $b_+(N) = 0$ & $\mathcal{Y}(N) \geq 0$

$$\mathcal{Y}(M \# N) = \mathcal{Y}(M).$$

- M, N: as above.
  Let $C_1 \subset M$ & $C_2 \subset N$ be circles s.t. $[C_2] \neq 0$ in $H_1(N; \mathbb{R})$.
  Let $n(C_i)$ be the tubular nbd of C_i.
  Let $\tilde{M} = (M \setminus n(C_1)) \underset{\partial n(C_1) = \partial n(C_2)}{\cup} (N \setminus n(C_2))$.

$$\mathcal{Y}(\tilde{M}) = \mathcal{Y}(M)$$

The proof uses the (ordinary) SW.

[LeBrun, '97, Gursky-LeBrun,'98]

$k = 1, 2, 3, \quad \forall l,$

$$\mathcal{Y}(\mathbb{C}\mathrm{P}^2 \# l(S^1 \times S^3)) = \mathcal{Y}(\mathbb{C}\mathrm{P}^2) = 12\sqrt{2}\pi < \mathcal{Y}(S^4) = 8\sqrt{6}\pi$$
$$0 < \mathcal{Y}(k\,\mathbb{C}\mathrm{P}^2 \# l(S^1 \times S^3)) \leq 4\pi\sqrt{2k+16} < \mathcal{Y}(S^4)$$

Proof uses

- perturbed SW eqns
- modified scalar curvature,
- conformal scaling trick,
- $\mathrm{ind}\, D_A > 0 \Rightarrow \exists \Phi$ s.t. $D_A \Phi = 0$ & Weitzenböck formula

Outline of the proof

Our main theorem, again.

Theorem 1 (Ishida-Matsuo-N.,'14)

Let M be a compact minimal Kähler surface, $b_+ \geq 2$, $c_1^2(M) \geq 0$.
Let $Z = Z_1 \# \cdots \# Z_k$ such that

$$Z_i = S^2 \times \Sigma \quad \text{or} \quad S^1 \times Y^3 \quad \text{or} \quad N$$

with $g(\Sigma) > 0$, $\mathcal{Y}(N) \geq 0$, $b_+(N) = 0$. (Ex. $N = \overline{\mathbb{C}\mathrm{P}^2}$)

Then $\mathcal{Y}(M \# Z) = \mathcal{Y}(M) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}$.

$$\mathcal{I}_s(M) := \inf_{g \in \mathcal{M}(M)} \int_M |s_g|^{\frac{n}{2}} d\mu_g$$

Theorem

- [Kobayashi,'90][Besson-Curtois-Gallot, '91]

$$\mathcal{I}_s(M) = \begin{cases} 0 & \text{if } \mathcal{Y}(M) \geq 0 \\ |\mathcal{Y}(M)|^{\frac{n}{2}} & \text{if } \mathcal{Y}(M) \leq 0 \end{cases}$$

- [Kobayashi,'87]

$$\mathcal{I}_s(M \# N) \leq \mathcal{I}_s(M) + \mathcal{I}_s(N)$$

For our $M\#Z = M\#Z_1\#\cdots\#Z_k$, if $\mathcal{Y}(M\#Z) \leq 0$, then

$$|\mathcal{Y}(M\#Z)|^2 = \mathcal{I}_s(M\#Z_1\#\cdots\#Z_k) \leq \mathcal{I}_s(M) + \mathcal{I}_s(Z_1) + \cdots + \mathcal{I}_s(Z_k).$$

- $\mathcal{I}_s(M) = 32\pi c_1^2(M)$ by [LeBrun]
- $S^2 \times \Sigma$ admits a PSC metric $\Rightarrow \mathcal{Y}(S^2 \times \Sigma) > 0$ $\Rightarrow \mathcal{I}_s(S^2 \times \Sigma) = 0.$
- $Z_i = S^1 \times Y$ or $N \Rightarrow \mathcal{Y}(Z_i) \geq 0 \Rightarrow \mathcal{I}_s(S^1 \times N) = 0.$

$$\boxed{\mathcal{Y}(M\#Z) \leq 0 \Rightarrow |\mathcal{Y}(M\#Z)|^2 = \mathcal{I}_s(M\#Z) \leq 32\pi c_1^2(M)}$$

Now, to prove is

$$\mathcal{Y}(M\#Z) \leq 0 \quad \& \quad |\mathcal{Y}(M\#Z)|^2 = \mathcal{I}_s(M\#Z) \geq 32\pi c_1^2(M)$$

These are proved by using $\mathrm{Pin}^-(2)$-monopole equations

$\mathrm{Pin}^{-}(2)$-monopole equations

- Seiberg-Witten equations are defined on a Spin^c-structure.
  ($\mathrm{U}(1)$-monopole equations)

$$\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$$

- $\mathrm{Pin}^{-}(2)$-monopole eqns are defined on a Spin^{c-}-structure.

$$\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^{-}(2)$$

$\mathrm{Spin}^{c-}(4)$

$$\mathrm{Pin}^-(2) = \langle \mathrm{U}(1), j \rangle = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}.$$

Two-to-one homomorphism $\mathrm{Pin}^-(2) \to \mathrm{O}(2)$

$$z \in \mathrm{U}(1) \subset \mathrm{Pin}^-(2) \mapsto z^2 \in \mathrm{U}(1) \cong \mathrm{SO}(2) \subset \mathrm{O}(2)$$
$$j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Definition $\mathrm{Spin}^{c-}(4) := \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$

- $\mathrm{Spin}^{c-}(4)/\mathrm{Pin}^-(2) = \mathrm{Spin}(4)/\{\pm 1\} = \mathrm{SO}(4)$
- $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}(4) = \mathrm{O}(2)$
- The id. compo. of $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$
  $= \mathrm{Spin}^c(4)$

  $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}^c(4) = \{\pm 1\}.$

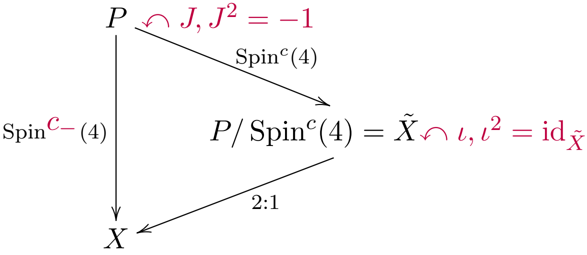
Spin^{c-}-structures

- X: an oriented Riemannian 4-manifold.
  $\longrightarrow Fr(X)$: The $\mathrm{SO}(4)$-frame bundle.
- $\tilde{X} \overset{2:1}{\to} X$: (nontrivial) double covering, $\ell := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$

[Furuta,08] A Spin^{c-}-structure $\mathfrak{s}$ on $\tilde{X} \to X$ is given by

- P: a $\mathrm{Spin}^{c-}(4)$-bundle over X,
- $P/\mathrm{Spin}^c(4) \overset{\cong}{\to} \tilde{X}$
- $P/\mathrm{Pin}^-(2) \overset{\cong}{\to} Fr(X)$.

- $E = P/\mathrm{Spin}(4) \overset{\mathrm{O}(2)}{\to} X$: characteristic $\mathrm{O}(2)$-bundle.
  $\longrightarrow$ ℓ-coefficient Euler class $\tilde{c}_1(E) \in H^2(X;\ell)$.
  $H^2(X;\ell) \overset{1:1}{\longleftrightarrow} \{\mathrm{O}(2)\text{-bundle } E \text{ over } X \text{ s.t. } E/\mathrm{SO}(2) \cong \tilde{X}\}/\text{iso}$.



- $P \xrightarrow{\mathrm{Spin}^c(4)} \tilde{X}$ defines a Spin^c-structure $\tilde{\mathfrak{s}}$ on $\tilde{X}$
- $J = [1, j] \in \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2) = \mathrm{Spin}^{c-}(4)$.
- Involution I on the spinor bundles $\tilde{S}^\pm$ of $\tilde{\mathfrak{s}}$:

$$\tilde{S}^\pm = P \times_{\mathrm{Spin}^c(4)} \mathbb{H}_\pm \curvearrowleft [J, j] =: I$$

$\Rightarrow$ $I^2 = 1$ & I is antilinear.

$\Rightarrow$ $S^\pm = \tilde{S}^\pm / I$ are the spinor bundles for the Spin^{c-}-str. $\mathfrak{s}$
$S^\pm$ are not complex bundles.

- Twisted Clifford multiplication

$$\rho\colon T^*X \otimes (\ell \otimes \sqrt{-1}\mathbb{R}) \to \mathrm{End}(S^+ \oplus S^-)$$

- An $\mathrm{O}(2)$-connection A on E + Levi-Civita $\Rightarrow$ Dirac operator

$$D_A\colon \Gamma(S^+) \to \Gamma(S^-)$$

- Weitzenböck formula

$$D_A^2\Phi = \nabla_A^*\nabla_A\Phi + \frac{s_g}{4}\Phi + \frac{\rho(F_A)}{2}\Phi$$

Remark The pull-back of A to π^*E has a (canonical) $\mathrm{U}(1)$-reduction $\tilde{A}$ on the determinant bundle of $\tilde{\mathfrak{s}}$.

$$D_A \cong \left(D_{\tilde{A}}\right)^I : \Gamma(\tilde{S^+})^I \to \Gamma(\tilde{S^+})^I$$

$\mathrm{Pin}^-(2)$-monopole equations

$$\begin{cases} D_A\Phi = 0, \\ \rho(F_A^+) = q(\Phi), \end{cases}$$

where

- A: $\mathrm{O}(2)$-connection on E & $\Phi \in \Gamma(S^+)$
- $F_A^+ \in \Omega^+(\ell \otimes \sqrt{-1}\mathbb{R})$
- $q(\Phi) = (\Phi^* \otimes \Phi) - \frac{1}{2}|\Phi|^2 \,\mathrm{id} \in \mathrm{End}(S^+)$

Remark

$\mathrm{Pin}^-(2)$-monopole on X = I-invariant Seiberg-Witten on $\tilde{X}$

- Max. principle + Weitzenböck
  $\Rightarrow$ No solution with $\Phi \not\equiv 0$ for a PSC metric.
- $b_+^\ell := \dim H_+(X; \ell \otimes \mathbb{R}) \geq 2$
  $\Rightarrow$ $\mathrm{Pin}^-(2)$-monopole invariant $\mathrm{SW}^{\mathrm{Pin}^-(2)}$ is defined.

\- Roughly, $\mathrm{SW}^{\mathrm{Pin}^-(2)} = \#\{$solutions with $\Phi \not\equiv 0$ $\}$ (mod 2)

Proposition 1 $\mathrm{SW}^{\mathrm{Pin}^-(2)} \neq 0 \Rightarrow \mathcal{Y}(M) \leq 0.$
$(\because \mathrm{SW}^{\mathrm{Pin}^-(2)} \neq 0 \Rightarrow$ No PSC metric $\Leftrightarrow \mathcal{Y}(M) \leq 0.)$

- Let $a \in \Omega^2(\ell \otimes \sqrt{-1}\mathbb{R})$ be the g-harmonic representative of $\tilde{c}_1(E)$.
- Decompose a into the g-self-dual & g-anti-self-dual parts:

$$a = a_+ + a_-$$

LeBrun's estimate

$$\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0 \Rightarrow \int_X |s_g|^2 d\mu_g \geq 32\pi^2 (a_+)^2 \quad \text{for } \forall g \in \mathcal{M}(X)$$

$(\because)$ The Weitzenböck formula.

$$|D_A\Phi|^2 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s_g|\Phi|^2 + \langle F_A^+ \cdot \Phi, \Phi\rangle.$$

$$(A,\Phi) : \text{a solution } \Rightarrow 0 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s_g|\Phi|^2 + |\Phi|^4$$

$$\int |\Phi|^4 d\mu \leq \int (-s_g)|\Phi|^2 d\mu \leq \left(\int |s_g|^2 d\mu\right)^{\frac{1}{2}} \left(\int |\Phi|^4 d\mu\right)^{\frac{1}{2}}$$

$$\int |s_g|^2 d\mu \geq \int |\Phi|^4 d\mu = 8\int |F_A^+|^2 d\mu = 32\pi^2 (a_+)^2.$$

Corollary $\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0 \Rightarrow \mathcal{I}_s(X) \geq 32\pi^2 a_+^2$

Non-vanishing theorem [N., Ishida-Matsuo-N.]

$\exists\,\mathrm{Spin}^{c-}$-structure on $M\#Z$ s.t.

- $\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0$
- $\forall g \quad (a_+)^2 \geq c_1^2(M)$
  where a_+ is the g-self-dual part of the g-harmonic form a representing $\tilde{c}_1(E)$.

From these, we obtain

$$\mathcal{Y}(M\#Z) \leq 0 \quad \& \quad |\mathcal{Y}(M\#Z)|^2 = \mathcal{I}_s(M\#Z) \geq 32\pi c_1^2(M)$$

Theorem 1 is proved.

On the Non-vanishing theorem

M: compact Kähler,
$Z = Z_1 \# \cdots \# Z_k$ where $Z_i = S^1 \times \Sigma$ or $S^1 \times Y$ or N
s.t. $g(\Sigma) \geq 1$, $b_+(N) = 0$

Non-vanishing theorem

$\exists\, \mathrm{Spin}^{c-}$-structure on $M \# Z$ s.t.

- $\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0$
- $\forall g \quad (a_+)^2 \geq c_1^2(M)$
  where a_+ is the g-self-dual part of the g-harmonic form a representing $\tilde{c}_1(E)$.

If $\exists Z_i = S^2 \times \Sigma$ or $S^1 \times Y$ with $b_1(Y) \geq 1$
$\Rightarrow$ SW & Donaldson inv. of $M \# Z$ are 0 ($\because b_+(M)$, $b_+(Z) \geq 1$.)

In general,

[Fact]

If $b_+(X)$, $b_+(Y) \geq 1$,
$\Rightarrow$ all of Donaldson inv & ordinary SW inv of $X\#Y$ are 0.

However, if $b_+(Y) = 0$, ordinary SW can be nontrivial.

[Fintushel-Stern, Kotschick-Morgan-Taubes, Ozsbath-Szabo, Froyshov]

- Y with $b_+(Y) = 0$.
- X: $\mathrm{SW}^{\mathrm{U}(1)}(X) \neq 0$

$\Rightarrow \mathrm{SW}^{\mathrm{U}(1)}(X\#Y) \neq 0$.

$\mathrm{SW}(X\#Y)$ can be calculated via gluing of solutions.
For example, assume Y has a PSC metric.

- When $\Phi \equiv 0$, SW eqn $\Leftrightarrow F_A^+ = 0$
- $b_+ > 0 \Rightarrow$ No solution on Y $\Rightarrow \mathrm{SW}(X\#Y) = 0$
- $b_+ = 0 \Rightarrow$ No solution with $\Phi \not\equiv 0$, but $\exists$solution with $\Phi \equiv 0$.
  $\Rightarrow \mathrm{SW}(X\#Y)$ can be nonzero

- $\mathrm{Pin}^{-}(2)$-monopole theory $=$ SW theory twisted along the local coefficient ℓ associated with the Spin^{c-}-structure.
- It can occur that $b_+^\ell = \dim H_+(X;\ell) = 0$, even if $b_+ \neq 0$.
- For $Z_i = S^2 \times \Sigma$ or $S^1 \times Y$, $\exists\, \mathrm{Spin}^{c-}$-structure s.t. $b_+^\ell = 0$.

$\Rightarrow$ We can prove $\mathrm{SW}^{\mathrm{Pin}^-(2)}(M\#Z) \neq 0$.

For future researches...

Problem Study X with $\mathcal{Y}(X) > 0$ by $\mathrm{Pin}^-(2)$-monopole.

Cf. [LeBrun, Gursky-LeBrun]

$$0 < \mathcal{Y}(\mathbb{C}\mathrm{P}^2) = \mathcal{Y}(\mathbb{C}\mathrm{P}^2 \# m(S^1 \times S^3)) = 12\sqrt{2}\pi < \mathcal{Y}(S^4) = 8\sqrt{6}\pi$$

WANTED
X: admitting a PSC metric & a loc. coeff. ℓ with $b_+^\ell = 1$, $b_-^\ell = 0$.