$\mathrm{Pin}^{-}(2)$-monopole equations and its applications

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- Let X be a closed oriented 4-manifold.

Intersection form

$$Q_X\colon H^2(X;\mathbb{Z})/\text{torsion} \times H^2(X;\mathbb{Z})/\text{torsion} \to \mathbb{Z},$$
$$(a,b) \mapsto \langle a \cup b, [X] \rangle.$$

- Q_X is a symmetric bilinear unimodular form.

[J.H.C.Whitehead '49]

If $\pi_1 X = 1$, the homotopy type of X is determined by the isomorphism class of Q_X.

In 4-dim. TOP

$\underline{\pi_1 X = 1}$

[Freedman '82]

The homeo type of X is determined by

- the iso. class of Q_X if Q_X is even,
- the iso. class of Q_X & $\mathrm{ks}(X)$ if Q_X is odd.

$\underline{\pi_1 X \neq 1}$

If $\pi_1 X$ is "Good" $\Rightarrow$ Freedman theory + Surgery theory.
$\to$ Difficult.

In 4-dim. DIFF

- Let X be a closed oriented smooth 4-manifold.

[Rohlin] If X is spin $\Rightarrow \mathrm{sign}(X) \equiv 0 \bmod 16$.

[Donaldson] If Q_X is definite $\Rightarrow Q_X \sim$ The diagonal form.

[Furuta] If X is spin & Q_X is indefinite, then

$$b_2(X) \geq \frac{10}{8}|\,\mathrm{sign}(X)| + 2.$$

Refinements, variants

[Furuta-Kametani '05]

The strong 10/8-inequality in the case when $b_1(X) > 0$.

[Froyshov '10]

A local coefficient analogue of Donaldson's theorem.

local coefficients $\leftrightarrow$ double coverings $\leftrightarrow$ $H^1(X;\mathbb{Z}/2)$

Froyshov's results

4-manifolds and intersection forms with local coefficients, arXiv:1004.0077

- Suppose a double covering $\tilde{X} \to X$ is given.
- $l := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, a $\mathbb{Z}$-bundle over X.
  $\longrightarrow H^*(X; l)$: l-coefficient cohomology.
- Note $l \otimes l = \mathbb{Z}$. The cup product

$$\cup \colon H^2(X; l) \times H^2(X; l) \to H^4(X; \mathbb{Z}) \cong \mathbb{Z},$$

  induces the intersection form with local coefficient

$$Q_{X,l} \colon H^2(X; l)/\text{torsion} \times H^2(X; l)/\text{torsion} \to \mathbb{Z}.$$

- $Q_{X,l}$ is also a symmetric bilinear unimodular form.

A special case of Froyshov's theorem

- X: a closed connected oriented smooth 4-manifold s.t.

$$b^+(X) + \dim_{\mathbb{Z}/2}(\text{tor} H_1(X; \mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2. \tag{1}$$

- $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

- The original form of Froyshov's theorem is:

  If X with $\partial X = Y : \mathbb{Z}HS^3$ satisfies (1)
  & $Q_{X,l}$ is nonstandard definite
  $\Rightarrow \delta_0 \colon HF^4(Y; \mathbb{Z}/2) \to \mathbb{Z}/2$ is non-zero.
- $Y = S^3 \Rightarrow HF^4(Y; \mathbb{Z}/2) = 0 \Rightarrow$The above result.

- The proof uses the moduli space of $\mathrm{SO}(3)$-instantons on a $\mathrm{SO}(3)$-bundle V.
- Twisted reducibles (stabilizer $\cong \mathbb{Z}/2$) play an important role.
  V is reduced to $\lambda \oplus E$, where E is an $\mathrm{O}(2)$-bundle,
  $\lambda = \det E$: a nontrivial $\mathbb{R}$-bundle.

Cf [Fintushel-Stern'84] gives an alternative proof of Donaldson's theorem by using $\mathrm{SO}(3)$-instantons.
$\longrightarrow$ Abelian reducibles (stabilizer $\cong \mathrm{U}(1)$)
V is reduced to $\underline{\mathbb{R}} \oplus L$, where L is a $\mathrm{U}(1)$-bundle.

- Donaldson's theorem is proved by Seiberg-Witten theory, too.

Question

Can we prove Froyshov's result by Seiberg-Witten theory?

$\longrightarrow$ Our result would be an answer.

Main results

Theorem 1.(N.)

- X: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

Cf. Froyshov's theorem

- X: — s.t. $b^+(X) + \dim_{\mathbb{Z}/2}(\mathrm{tor} H_1(X;\mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2$.
- $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

Main results

Theorem 1.(N.)

- X: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow$ $Q_{X,l} \sim$ diagonal.

- For the proof, we will introduce a variant of Seiberg-Witten equations
  $\longrightarrow$ $\mathrm{Pin}^-(2)$-monopole equations on Spin^{c-}-structures on X.
- Spin^{c-}-structure is a $\mathrm{Pin}^-(2)$-variant of Spin^c-str. defined by M.Furuta, whose complex structure is "twisted along l".

- The moduli space of $\mathrm{Pin}^-(2)$-monopoles is compact.
  $\longrightarrow$ Bauer-Furuta theory can be developed.

Furuta's theorem

Let X be a closed ori. smooth spin 4-manifold with indefinite Q_X.

$$b_+(X) \geq -\frac{\mathrm{sign}(X)}{8} + 1.$$

Theorem 2(N.)

Let X be a closed connected ori. smooth 4-manifold. For any nontrivial $\mathbb{Z}$-bundle $l \to X$ s.t. $w_1(\lambda)^2 = w_2(X)$, where $\lambda = l \otimes \mathbb{R}$,

$$b_+(X;\lambda) \geq -\frac{\mathrm{sign}(X)}{8},$$

where $b_+(X;\lambda) = \mathrm{rank}\, H^+(X;\lambda)$.

Applications

Recall fundamental theorems.

1. [Rohlin] X^4: closed spin $\Rightarrow \mathrm{sign}(X) \equiv 0$ mod 16.
2. [Donaldson] Definite $\Rightarrow$ diagonal.
3. [Furuta] The $10/8$-inequality

3' [Furuta-Kametani] The strong $10/8$-inequality in the case when $b_1 > 0$.

Corollary 1(N.)

$\exists$ Nonsmoothable closed indefinite spin 4-manifolds satisfying

- $\mathrm{sign}(X) \equiv 0$ mod 16,
- the strong $10/8$-inequality.

Proof

- Let M be T^4 or $T^2 \times S^2$. $\Rightarrow Q_{T^4} = 3H$, $Q_{T^2\times S^2} = H$.
- If $l' \to M$ is any nontrivial $\mathbb{Z}$-bundle, $\Rightarrow b_2(M; l') = 0$ & $w_1(l' \otimes \mathbb{R})^2 = 0$.
- Let V be a topological 4-manifold s.t. $\pi_1 V = 1$, Q_V is even and definite, $\mathrm{sign}(V) \equiv 0$ mod 16. ($\Rightarrow$ V is spin.)
- Choose a large k s.t. $X = V\#kM$ satisfies the strong $10/8$-inequality.
- Let $l := \underline{\mathbb{Z}}\#kl' \to X$. $\Rightarrow Q_{X,l} = Q_V$, $w_1(l\otimes\mathbb{R})^2 = 0$.
- Suppose X is smooth. By Theorem 1, $Q_{X,l} = Q_V \sim$ diagonal. Contradiction.

Remark

Similar examples can be constructed by using Theorem 2.

Non-spin manifolds

10/8-conjecture

Every non-spin closed smooth 4-manifold X with even form satisfies

$$b_2(X) \geq \frac{10}{8}|\operatorname{sign}(X)|.$$

[Bohr,'02],[Lee-Li,'00]

If the 2-torsion part of $H_1(X;\mathbb{Z})$ is $\mathbb{Z}/2^i$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\Rightarrow$ the 10/8-conjecture is true.

Corollary 2(N.)

$\exists$ Nonsmoothable non-spin 4-manifolds X with even form s.t.

- the 2-torsion part of $H_1(X;\mathbb{Z}) \cong \mathbb{Z}/2$,
- the 10/8-conjecture is true.

Review on the Seiberg-Witten theory

- X: a closed ori. smooth 4-manifold with a Rimaniann metric.
- Suppose a Spin^c-structure on X is given.
  $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$
  $\rightarrow L$: the determinant $\mathrm{U}(1)$-bundle.
- Monopole map

$$\mu_{SW} \colon \mathcal{A}(L) \times \Gamma(S^+) \to \Omega^+ \times \Gamma(S^-),$$

  where $\mathcal{A}(L)$: the space of $\mathrm{U}(1)$-connections on L,
  $S^{\pm}$: spinor bundles.
- solutions of SW-eqn $\leftrightarrow$ zero points of μ_{SW}.
- μ_{SW} is $\mathcal{G}_{SW}$-equivariant, where $\boxed{\mathcal{G}_{SW} = \mathrm{Map}(X, \mathrm{U}(1)).}$

- The moduli space $\mu_{SW}^{-1}(0)/\mathcal{G}_{SW}$.
- Restriction to intersection forms
- SW-invariants $\in \mathbb{Z}$
- Bauer-Furuta invariants $\in$ a stable cohomotopy group

Overview of $\mathrm{Pin}^-(2)$-monopole theory

- Spin^{c-}-structure on X
  $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$ $(\mathrm{Pin}^-(2) = U(1) \cup j\,\mathrm{U}(1))$
  $\rightarrow E$: $\mathrm{O}(2)$-bundle
- $\mathrm{Pin}^-(2)$-monopole map

$$\mu\colon \mathcal{A}(E) \times \Gamma(S^+) \rightarrow \Omega^+(i\lambda) \times \Gamma(S^-),$$

  where $\mathcal{A}(E)$: the space of $\mathrm{O}(2)$-connections on E,
  $S^\pm$: spinor bundles,
  $\lambda = \det E$.
- μ is $\mathcal{G}$-equivariant, where

$$\boxed{\mathcal{G} = \Gamma(E \times_{\mathrm{O}(2)} \mathrm{U}(1)).}$$

  where $\mathrm{O}(2) \rightarrow \{\pm 1\} \curvearrowright \mathrm{U}(1)$ by $z \mapsto z^{-1}$.

- The moduli space $\mu^{-1}(0)/\mathcal{G}$.
- Restriction to intersection forms with local coefficients
  $\to$ Today's topic
- $\mathrm{Pin}^-(2)$-monopole SW-inv. $\in \mathbb{Z}_2$ or $\mathbb{Z}$
- $\mathrm{Pin}^-(2)$-monopole BF-inv. $\in$ a stable cohomotopy group

$\mathrm{Spin}^{c-}(n)$-groups

$$\mathrm{Pin}^-(2) = \langle \mathrm{U}(1), j\rangle = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}.$$

The two-to-one homomorphism $\mathrm{Pin}^-(2) \to \mathrm{O}(2)$ is defined by

$$z \in \mathrm{U}(1) \subset \mathrm{Pin}^-(2) \mapsto z^2 \in \mathrm{U}(1) \subset \mathrm{O}(2),$$
$$j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition $\mathrm{Spin}^{c-}(n) := \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$.

$$1 \to \{\pm 1\} \to \mathrm{Spin}^{c-}(n) \to \mathrm{SO}(n) \times \mathrm{O}(2) \to 1.$$

Note. The id. compo. of $\mathrm{Spin}^{c-}(n)$ is

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{U}(1).$$

Spin^{c-}-structures

- Let X be an oriented 4-manifold.
- Fix a Riemannian metric.
  $\longrightarrow Fr(X)$: The $\mathrm{SO}(4)$-frame bundle.

Spin^{c-}-structure

A Spin^{c-}-structure on X is given by (P, τ) s.t.

- P: a $\mathrm{Spin}^{c-}(4)$-bundle over X,
- $\tau\colon P/\mathrm{Pin}^-(2) \xrightarrow{\cong} Fr(X)$.

Then we have

- $E = P/\mathrm{Spin}(4)$: $\mathrm{O}(2)$-bundle over X,
- $\tilde{X} = P/\mathrm{Spin}^c(4)$: a double covering of X.

$$\det E \cong \tilde{X} \times_{\{\pm 1\}} \mathbb{R} =: \lambda.$$

The following data $(\tilde{P}, \tilde{\tau}, \tilde{\imath})$ on $\tilde{X}$ gives a Spin^{c-}-structure on X.

- $(\tilde{P}, \tilde{\tau})$: A Spin^c-structure on $\tilde{X}$.
  - $\tilde{P}$: a $\mathrm{Spin}^c(4)$-bundle over $\tilde{X}$,
  - $\tilde{\tau}\colon \tilde{P}/\mathrm{U}(1) \xrightarrow{\cong} Fr(\tilde{X})$
- $\tilde{\imath}\colon \tilde{P} \to \tilde{P}$: a map covering $\tilde{X} \xrightarrow{(-1)} \tilde{X}$ s.t.

$$\tilde{\imath}(pz) = \tilde{\imath}(p)z^{-1}, \text{ for } z \in \mathrm{U}(1), \text{ and } \tilde{\imath}^2 = -1.$$

Let $\Delta^\pm$ be the complex spinor rep. of $\mathrm{Spin}^c(4)$.
$\Rightarrow \exists j$-action on $\Delta^\pm$ s.t.

$$j^2 = -1, \text{ and } jz = z^{-1}j \text{ for } z \in \mathrm{U}(1)$$

$\Rightarrow I = (\tilde{\imath}, j)$ is an antilinear involution on $\tilde{S}^\pm = \tilde{P} \times_{\mathrm{Spin}^c(4)} \Delta^\pm$.
$\Rightarrow S^\pm = \tilde{S}^\pm/I$ over X is the spinor bundle for the Spin^{c-}-str.
$\Rightarrow S^\pm$ are NOT complex bundles.

Take the I-invariant part of the monopole map μ_{SW} on $\tilde{X}$.
$\Rightarrow \mathrm{Pin}^-(2)$-monopole map,

$$\mu\colon \mathcal{A}\times\Gamma(S^+)\to\Omega^+(i\lambda)\times\Gamma(S^-),$$

where $\lambda=\tilde{X}\times_{\{\pm1\}}\mathbb{R}$,
$\mathcal{A}=\{\mathrm{O}(2)\text{-connections on } E\}\leftarrow$ an affine sp. of $\Omega^1(i\lambda)$

Symmetry

$$\begin{aligned}\mathcal{G}=&\{f\in\mathrm{Map}(\tilde{X},\mathrm{U}(1))\,|\,f(-x)=f(x)^{-1}\}\\=&\Gamma(\tilde{X}\times_{\{\pm1\}}\mathrm{U}(1)),\end{aligned}$$

where $\{\pm1\}\curvearrowright\mathrm{U}(1)$ by $z\mapsto z^{-1}$.

- A basic fact due to [Furuta '08]:
  $\exists\ \mathrm{Spin}^{c-}$-structure on $(X,E)\Leftrightarrow w_2(X)=w_2(E)+w_1(E)^2$.
- If $E\cong\underline{\mathbb{R}}\oplus\lambda=\underline{\mathbb{R}}\oplus(\tilde{X}\times_{\{\pm1\}}\mathbb{R})$
  $\Rightarrow P$ is reduced to $\mathrm{Spin}(4)\times_{\{\pm1\}}\langle\pm1,\pm j\rangle$-bundle
  $\Rightarrow$ an analogy of spin structure.
  $\Rightarrow\exists$Larger symmetry

  $$\mathcal{G}'=\Gamma(\tilde{X}\times_{\{\pm1\}}\mathrm{Pin}^-(2)),$$

  where $\{\pm1\}\curvearrowright\mathrm{Pin}^-(2)=\mathrm{U}(1)\cup j\,\mathrm{U}(1)$ is given by

  $$\begin{aligned}z\mapsto&z^{-1}\quad\text{for } z\in\mathrm{U}(1),\\j\mapsto&j.\end{aligned}$$

Moduli spaces

$$\mathcal{M} = \mu^{-1}(0)/\mathcal{G} \subset (\mathcal{A} \times \Gamma(S^+))/\mathcal{G}$$

Proposition

- $\mathcal{M}$ is compact.
- The virtual dimension of $\mathcal{M}$:

$$d = \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b_0(X;l) - b_1(X;l) + b_+(X;l)),$$

where $\tilde{c}_1(E)$ is the *twisted 1st Chern class*.

- $\tilde{c}_1(E)$ is the Euler class of E considered in $H^2(X;l)$ where $l \subset \lambda = \det E$, sub-$\mathbb{Z}$-bundle.
- If l is nontrivial & X connected $\Rightarrow b_0(X;l) = 0$.

Reducibles

- For $(A, \phi) \in \mathcal{A} \times \Gamma(S^+)$, if $\phi \neq 0 \Rightarrow \mathcal{G}$-action is free.
- The stabilizer of $(A, 0) = \{\pm 1\} \subset \mathcal{G} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1))$.
- The elements of the form $(A, 0)$ are called reducibles.
- In general, $\{$ reducible solutions $\}/\mathcal{G} \cong T^{b_1(X;l)} \subset \mathcal{M}$.

Cf. In the SW-case, the stabilizer of $(A, 0) = S^1 \subset \mathrm{Map}(X, S^1)$.

Key difference

Ordinary SW case

- Reducible $\rightarrow$ The stabilizer $= S^1$.

$$\mathcal{M}_{SW} \setminus \{\text{reducibles}\} \subset (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G}_{SW} \simeq B\mathcal{G}_{SW}$$
$$\simeq T^{b_1(X)} \times \mathbb{C}\mathrm{P}^\infty .$$

$\mathrm{Pin}^-(2)$-monopole case

- Reducible $\rightarrow$ The stabilizer $= \{\pm 1\}$.

$$\mathcal{M} \setminus \{\text{reducibles}\} \subset (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G} \simeq B\mathcal{G}$$
$$\simeq T^{b_1(X;l)} \times \mathbb{R}\mathrm{P}^\infty .$$

Proof of Theorem 1

Outline of the proof

- We will prove every characteristic element w of $Q_{X,l}$ satisfies
$$|w^2| \geq \operatorname{rank} H^2(X;l),$$
by proving for every E,
$$d = \dim \mathcal{M} \leq 0.$$
- Then Elkies' theorem implies $Q_{X,l}$ should be standard.

- An element w in a unimodular lattice L is called *characteristic* if $w \cdot v \equiv v \cdot v \bmod 2$ for $\forall v \in L$.

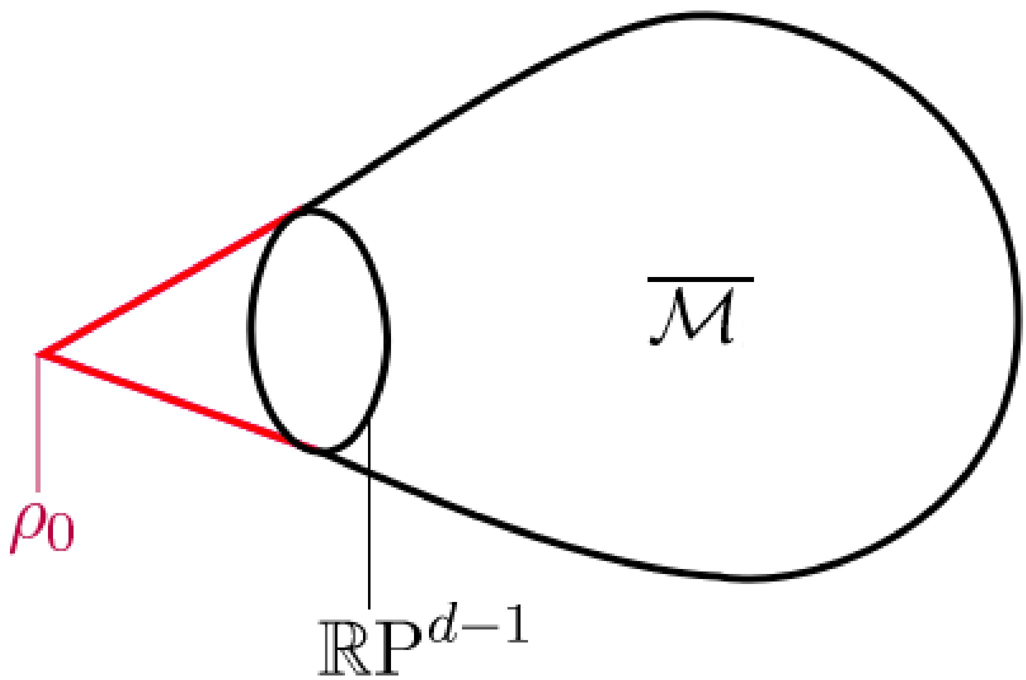
[Elkies '95]

$L \subset \mathbb{R}^n$: unimodular lattice. If $\forall$characteristic element $w \in L$ satisfies $|w^2| \geq \operatorname{rank} L$, $\Rightarrow L \cong$ diagonal.

The structure of $\mathcal{M}$ when $b_+(X;l) = 0$

- Suppose a Spin^{c-}-structure (P, τ) on X is given.
- For simplicity, assume $b_1(X, l) = 0$.
  $\Rightarrow \exists^1$ reducible class $\rho_0 \in \mathcal{M}$.
- Perturb the $\mathrm{Pin}^-(2)$-monopole equations by adding $\eta \in \Omega^+(i\lambda)$ to the curvature equation. $\to F_A^+ = q(\phi) + \eta$.
- For generic η, $\mathcal{M} \setminus \{\rho_0\}$ is a d-dimensional manifold.
- Fix a small neighborhood $N(\rho_0)$ of $\{\rho_0\}$.
  $\Rightarrow N(\rho_0) \cong \mathbb{R}^d/\{\pm 1\} =$ a cone of $\mathbb{RP}^{d-1}$

  Then $\overline{\mathcal{M}} := \overline{\mathcal{M} \setminus N(\rho_0)}$ is a compact d-manifold & $\partial\overline{\mathcal{M}} = \mathbb{RP}^{d-1}$.



- Note $\overline{\mathcal{M}} \subset (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G} =: \mathcal{B}^*$.
- Recall $\mathcal{B}^* \underset{h.e.}{\simeq} T^{b_1(X;l)} \times \mathbb{R}\mathrm{P}^\infty$.

Lemma

If $b_+(X;l) = 0$ & $b_1(X;l) = 0 \Rightarrow d = \dim \mathcal{M} \leq 0$.

Proof

- Suppose $d > 0$.
- Recall $\overline{\mathcal{M}}$ is a compact d-manifold s.t. $\partial\overline{\mathcal{M}} = \mathbb{R}\mathrm{P}^{d-1}$.
- $\exists C \in H^{d-1}(\mathcal{B}^*;\mathbb{Z}/2) \cong H^{d-1}(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$ s.t. $\langle C, [\partial\overline{\mathcal{M}}]\rangle \neq 0$. $\Rightarrow$ Contradiction.

- Note $\mathrm{sign}(X) = b_+(X;l) - b_-(X;l)$ for any $\mathbb{Z}$-bundle l.
- By Lemma, if l is nontrivial & $b_+(X;l) = 0$ & $b_1(X;l) = 0$,

$$\begin{aligned} d =& \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b_0(X;l) - b_1(X;l) + b_+(X;l)) \\ =& \frac{1}{4}(\tilde{c}_1(E)^2 + b_2(X;l)) \leq 0. \end{aligned}$$

Note $\tilde{c}_1(E)^2 \leq 0$ if $b_+(X;l) = 0$.

- Therefore, for any E which admits a Spin^{c-}-structure,

$$b_2(X;l) \leq |\tilde{c}_1(E)^2|.$$

By varying E, we can prove every characteristic element w satisfies

$$b_2(X;l) \leq |w^2|.$$

The outline of the proof of Theorem 2

- If $E = \underline{\mathbb{R}} \oplus \lambda \Rightarrow \mathrm{Spin}^{c-}$-structure on (X, E) has the larger symmetry $\mathcal{G}' = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{Pin}^-(2))$.
- For simplicity, assume $b_1(X; l) = 0$.
- Then, by taking finite dimensional approximation of the monopole map, we obtain a proper $\mathbb{Z}_4$-equivariant map

$$f\colon \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

  where
  - $\tilde{\mathbb{R}}$ is $\mathbb{R}$ on which $\mathbb{Z}_4$ acts via $\mathbb{Z}_4 \to \mathbb{Z}_2 = \{\pm 1\} \curvearrowright \mathbb{R}$,
  - $\mathbb{C}_1$ is $\mathbb{C}$ on which $\mathbb{Z}_4$ acts by multiplication of i,
  - $k = -\operatorname{sign}(X)/8$, $b = b_+(X; \lambda)$, m, n are some integers.

  Here, $\mathbb{Z}_4$ is generated by the constant section

$$j \in \mathcal{G}' = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{Pin}^-(2)).$$

- By using the techniques of equivariant homotopy theory, e.g., tom Dieck's character formula, we can see that any proper $\mathbb{Z}_4$-map of the form,

$$f\colon \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

  should satisfy $b \geq k$.
- That is,

$$b_+(X; \lambda) \geq -\frac{1}{8} \operatorname{sign}(X).$$

Finite dimensional approximation

- Take a flat connection A_0 on $\underline{\mathbb{R}} \oplus \lambda$.

$\mathrm{Pin}^-(2)$-monopole map

$$\mu\colon \Omega^1(i\lambda) \oplus \Gamma(S^+) \to (\Omega^0 \oplus \Omega^+)(i\lambda) \oplus \Gamma(S^-) =: \mathcal{W},$$
$$(a, \phi) \mapsto (d^*a, F_{A_0} + d^+a + q(\phi), D_{A_0+a}\phi).$$

- Let $l(a, \phi) := (d^*a, d^+a, D_{A_0}\phi)$ be the linear part of μ.
  $\to$ l is Fredholm.
- $c = \mu - l$: quadratic, compact.
- Choose a finite dim. subspace $U \subset \mathcal{W}$ s.t. $\dim U \gg 1$,
  $U \supset (\operatorname{im} l)^\perp$
- Let $V := l^{-1}(U)$ & $p\colon \mathcal{W} \to U$ be the L^2-projection.
- Define $f\colon V \to U$ by $\boxed{f = l + pc.}$ $\to$ f: proper, $\mathbb{Z}_4$-equiv.

The genus of embedded surfaces in 4-manifolds

Theorem

- X: closed ori. 4-manifold
- c: Spin^c-structure on X.
  L: the determinant line bundle of c.
- $\Sigma \subset X$: connected embedded surface
  s.t. $[\Sigma] \in H_2(X; \mathbb{Z})$, $[\Sigma]\cdot[\Sigma] \geq 0$.

If $\mathrm{SW}(X, c) \neq 0$ or $\mathrm{BF}(X, c) \neq 0$, then

$$-\chi(\Sigma) = 2g - 2 \geq c_1(L)[\Sigma] + [\Sigma]\cdot[\Sigma].$$

- This is due to: [Kronheimer-Mrowka], [Fintushel-Stern], [Morgan-Szabo-Taubes], [Ozsvath-Szabo], [Furuta-Kametani-Matsue-Minami]...

The genus of nonorientable embedded surfaces

- (X^4, l) as before.

Let us consider a connected surface Σ s.t.

- $i\colon \Sigma \hookrightarrow X$: embedding
- The orientation coefficient of $\Sigma = i^* l$

$\to$ $\exists$Fundamental class $[\Sigma] \in H_2(\Sigma; i^* l)$.
Let $\alpha := i_*[\Sigma] \in H_2(X; l)$, where $i_*\colon H_2(\Sigma; i^* l) \to H_2(X; l)$.

Proposition

For $\forall \alpha \in H_2(X; l)$, there exists Σ as above.

Remark

- Σ may be orientable or nonorientable.

Theorem 3.(N.)

- (X, l, Σ) as above.
- Let $\alpha := i_*[\Sigma] \in H^2(X; l)$. Suppose $\alpha \cdot \alpha \geq 0$.
- c: Spin^{c-}-structure s.t. $\det E = l \otimes \mathbb{R}$.
- $\tilde{c}$: the Spin^c-structure on $\tilde{X}$ induced from c.

If $\mathrm{SW}^{\mathrm{Pin}}(X, c) \neq 0$ or $\mathrm{BF}^{\mathrm{Pin}}(X, c) \neq 0$
or $\mathrm{SW}(\tilde{X}, \tilde{c}) \neq 0$ or $\mathrm{BF}(\tilde{X}, \tilde{c}) \neq 0$, then

$$-\chi(\Sigma) \geq \tilde{c}_1(E) \cdot \alpha + \alpha \cdot \alpha$$

Remark

- Σ: orientable $\Rightarrow -\chi(\Sigma) = 2g - 2$.
- Σ: nonorientable $\Rightarrow -\chi(\Sigma) = g - 1$.

Example

- X: Enriques surface
  $\Rightarrow \pi_1 X = \mathbb{Z}/2$
  $\Rightarrow \tilde{X} = K3$, $l := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, $\lambda := \tilde{X} \times_{\{\pm 1\}} \mathbb{R}$
  $\Rightarrow \exists c$: Spin^{c-}-structure s.t. $E \cong \underline{\mathbb{R}} \oplus \lambda$. $(\tilde{c}_1(E) = 0)$
  $\Rightarrow \mathrm{SW}^{\mathrm{Pin}}(X, c) \neq 0$ $(\mathrm{SW}(\tilde{X}, \tilde{c}) \neq 0)$

For $\Sigma \stackrel{i}{\hookrightarrow} X$ s.t. $\alpha = i_*[\Sigma] \in H_2(X; l)$ & $\alpha \cdot \alpha \geq 0$

$$-\chi(\Sigma) \geq \alpha \cdot \alpha.$$

Final remarks

- $\mathrm{Pin}^-(2)$-monopole invariants
  - Calculation, gluing formula, stable cohomotopy refinements
- When $\tilde{X}$: symplectic & $I^*\omega = -\omega$,

  $$\mathrm{Pin}^-(2)\text{-monopole inv.} \underset{??}{=} \text{real Gromov-Witten inv.}$$

  Cf. [Tian-Wang]
- $\mathrm{Pin}^-(2)$-monopole Floer theory?
  $\mathrm{Pin}^-(2)$ Heegaard Floer theory?
- "Witten conjecture" for $\mathrm{Pin}^-(2)$-monopole invariants?
  - [Feehan-Leness] SW = Donaldson

    $\mathrm{Pin}^-(2)$-monopole inv. $=$???