$\mathrm{Pin}^{-}(2)$-monopole equations and intersection forms with local coefficients of 4-manifolds

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- Let X be a closed oriented 4-manifold.

Topological invariants for X

- $\pi_1 X$, cohomology ring, k-invariants...

Intersection form

$$Q_X \colon H^2(X;\mathbb{Z})/\text{torsion} \times H^2(X;\mathbb{Z})/\text{torsion} \to \mathbb{Z},$$
$$(a,b) \mapsto \langle a \cup b, [X] \rangle.$$

- Q_X is a symmetric bilinear unimodular form.

[J.H.C.Whitehead '49]

If $\pi_1 X = 1$, the homotopy type of X is determined by the isomorphism class of Q_X.

In 4-dim. TOP

$\pi_1 X = 1$

[Freedman '82]

The homeo type of X is determined by

- the iso. class of Q_X if Q_X is even,
- the iso. class of Q_X & $\mathrm{ks}(X)$ if Q_X is odd.

$\pi_1 X \neq 1$

If $\pi_1 X$ is "Good" $\Rightarrow$ Freedman theory + Surgery theory.
$\to$ Difficult.

In 4-dim. DIFF

- Let X be a closed oriented smooth 4-manifold.

[Rohlin] If X is spin $\Rightarrow \mathrm{sign}(X) \equiv 0$ mod 16.

[Donaldson] If Q_X is definite $\Rightarrow Q_X \sim$ The diagonal form.

[Furuta] If X is spin & Q_X is indefinite, then

$$b_2(X) \geq \frac{10}{8}|\,\mathrm{sign}(X)| + 2.$$

Refinements, variants

[Furuta-Kametani '05]

The strong $10/8$-inequality in the case when $b_1(X) > 0$.

[Froyshov '10]

A local coefficient analogue of Donaldson's theorem.

local coefficients $\leftrightarrow$ double coverings $\leftrightarrow H^1(X;\mathbb{Z}/2)$

Froyshov's results

4-manifolds and intersection forms with local coefficients, arXiv:1004.0077

- Suppose a double covering $\tilde{X} \to X$ is given.
- $l := \tilde{X} \times_{\mathbb{Z}_2} \mathbb{Z}$, a $\mathbb{Z}$-bundle over X.
  $\longrightarrow H^*(X; l)$: l-coefficient cohomology.
- Note $l \otimes l = \mathbb{Z}$. The cup product

$$\cup\colon H^2(X; l) \times H^2(X; l) \to H^4(X; \mathbb{Z}) \cong \mathbb{Z},$$

  induces the intersection form with local coefficient

$$Q_{X,l}\colon H^2(X; l)/\text{torsion} \times H^2(X; l)/\text{torsion} \to \mathbb{Z}.$$

- $Q_{X,l}$ is also a symmetric bilinear unimodular form.

A special case of Froyshov's theorem

- X: a closed connected oriented smooth 4-manifold s.t.

$$b^+(X) + \dim_{\mathbb{Z}/2}(\text{tor} H_1(X; \mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2. \qquad (1)$$

- $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

- The original form of Froyshov's theorem is:
  If X with $\partial X = Y$: $\mathbb{Z}HS^3$ satisfies (1)
  & $Q_{X,l}$ is nonstandard definite
  $\Rightarrow \delta_0\colon HF^4(Y; \mathbb{Z}/2) \to \mathbb{Z}/2$ is non-zero.
- $Y = S^3 \Rightarrow HF^4(Y; \mathbb{Z}/2) = 0 \Rightarrow$The above result.

- The proof uses the moduli space of $\mathrm{SO}(3)$-instantons on a $\mathrm{SO}(3)$-bundle V.
- Twisted reducibles (stabilizer $\cong \mathbb{Z}/2$) play an important role.
  V is reduced to $\lambda \oplus E$, where E is an $\mathrm{O}(2)$-bundle,
  $\lambda = \det E$: nontrivial.

Cf [Fintushel-Stern'84] gives an alternative proof of Donaldson's theorem by using $\mathrm{SO}(3)$-instantons.
$\longrightarrow$ Abelian reducibles (stabilizer $\cong \mathrm{U}(1)$)
V is reduced to $\underline{\mathbb{R}} \oplus L$, where L is a $\mathrm{U}(1)$-bundle.

- Donaldson's theorem is proved by Seiberg-Witten theory, too.

Question

Can we prove Froyshov's result by Seiberg-Witten theory?
$\longrightarrow$ Our result would be an answer.

Main results

Theorem 1.(N.)

- X: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

Cf. Froyshov's theorem

- X: — s.t. $b^+(X) + \dim_{\mathbb{Z}/2}(\mathrm{tor} H_1(X;\mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2$.
- $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

Main results

Theorem 1.(N.)

- X: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

- For the proof, we will introduce a variant of Seiberg-Witten equations
  $\longrightarrow$ $\mathrm{Pin}^-(2)$-monopole equations on Spin^{c-}-structures on X.
- Spin^{c-}-structure is a $\mathrm{Pin}^-(2)$-variant of Spin^c-str. defined by M.Furuta, whose complex structure is "twisted along l".

- The moduli space of $\mathrm{Pin}^-(2)$-monopoles is compact.
  $\longrightarrow$ Bauer-Furuta theory can be developed.

Furuta's theorem

Let X be a closed ori. smooth spin 4-manifold with indefinite Q_X.

$$b_+(X) \geq -\frac{\mathrm{sign}(X)}{8} + 1.$$

Theorem 2(N.)

Let X be a closed connected ori. smooth 4-manifold. For any nontrivial $\mathbb{Z}$-bundle $l \to X$ s.t. $w_1(\lambda)^2 = w_2(X)$, where $\lambda = l \otimes \mathbb{R}$,

$$b_+(X;\lambda) \geq -\frac{\mathrm{sign}(X)}{8},$$

where $b_+(X;\lambda) = \mathrm{rank}\, H^+(X;\lambda)$.

Applications

Recall fundamental theorems.

1. [Rohlin] X^4: closed spin $\Rightarrow \mathrm{sign}(X) \equiv 0$ mod 16.
2. [Donaldson] Definite $\Rightarrow$ diagonal.
3. [Furuta] The $10/8$-inequality

3' [Furuta-Kametani] The strong $10/8$-inequality in the case when $b_1 > 0$.

Corollary 1(N.)

$\exists$ Nonsmoothable closed indefinite spin 4-manifolds satisfying

- $\mathrm{sign}(X) \equiv 0$ mod 16,
- the strong $10/8$-inequality.

Proof

- Let M be T^4 or $T^2 \times S^2$. $\Rightarrow Q_{T^4} = 3H$, $Q_{T^2 \times S^2} = H$.
- If $l' \to M$ is any nontrivial $\mathbb{Z}$-bundle, $\Rightarrow b_2(M; l') = 0$ & $w_1(l' \otimes \mathbb{R})^2 = 0$.
- Let V be a topological 4-manifold s.t. $\pi_1 V = 1$, Q_V is even and definite, $\mathrm{sign}(V) \equiv 0$ mod 16. ($\Rightarrow V$ is spin.)
- Choose a large k s.t. $X = V \# kM$ satisfies the strong $10/8$-inequality.
- Let $l := \underline{\mathbb{Z}} \# kl' \to X$. $\Rightarrow Q_{X,l} = Q_V$, $w_1(l \otimes \mathbb{R})^2 = 0$.
- Suppose X is smooth. By Theorem 1, $Q_{X,l} = Q_V \sim$ diagonal. Contradiction.

Remark

Similar examples can be constructed by using Theorem 2.

Non-spin manifolds

10/8-conjecture

Every non-spin closed smooth 4-manifold X with even form satisfies

$$b_2(X) \geq \frac{10}{8}|\operatorname{sign}(X)|.$$

[Bohr,'02],[Lee-Li,'00]

If the 2-torsion part of $H_1(X;\mathbb{Z})$ is $\mathbb{Z}/2^i$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\Rightarrow$ the 10/8-conjecture is true.

Corollary 2(N.)

$\exists$ Nonsmoothable non-spin 4-manifolds X with even form s.t.

- the 2-torsion part of $H_1(X;\mathbb{Z}) \cong \mathbb{Z}/2$,
- the 10/8-conjecture is true.

The outline of the proof of Theorem 1

- The proof of Theorem 1 is almost parallel to the SW-proof of Donaldson's theorem.
- By using $\mathrm{Pin}^-(2)$-monopole moduli, we will prove every characteristic element w of $Q_{X,l}$ satisfies $|w^2| \geq \operatorname{rank} H^2(X;l)$. $\leftrightarrow$ (The dim. of the moduli) ≤ 0
- Then Elkies' theorem implies $Q_{X,l}$ should be standard.

- An element w in a unimodular lattice L is called *characteristic* if $w \cdot v \equiv v \cdot v$ mod 2 for $\forall v \in L$.

[Elkies '95]

If every characteristic element $w \in L$ satisfies $|w^2| \geq \operatorname{rank} L$, then $L \cong$ diagonal.

$\mathrm{Pin}^-(2)$-monopole equations

$$\mathrm{Pin}^-(2) = \langle \mathrm{U}(1), j\rangle = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}.$$

The two-to-one homomorphism $\mathrm{Pin}^-(2) \to \mathrm{O}(2)$ is defined by

$$z \in \mathrm{U}(1) \subset \mathrm{Pin}^-(2) \mapsto z^2 \in \mathrm{U}(1) \subset \mathrm{O}(2),$$
$$j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition $\mathrm{Spin}^{c-}(n) := \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$.

$$1 \to \{\pm 1\} \to \mathrm{Spin}^{c-}(n) \to \mathrm{SO}(n) \times \mathrm{O}(2) \to 1.$$

Cf. $\mathrm{Spin}^{c}(n) = \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{U}(1)$.

Spin^{c-}-structures

- Let X be an oriented n-manifold.
- Fix a Riemannian metric.
  $\longrightarrow F(X)$: The $\mathrm{SO}(n)$-frame bundle.
- Suppose an $\mathrm{O}(2)$-bundle E over X is given.

Spin^{c-}-structure

A Spin^{c-}-structure on (X, E) is given by (P, τ) s.t.

- P: a $\mathrm{Spin}^{c-}(n)$-bundle over X,
- $\tau\colon P/\{\pm 1\} \xrightarrow{\cong} F(X) \times_X E$.

Proposition(Furuta '08)

$\exists\, \mathrm{Spin}^{c-}$-structure on $(X, E) \Leftrightarrow w_2(X) = w_2(E) + w_1(E)^2$.

The case when $n = 4$

- $\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$.
- $\mathrm{Spin}^{c-}(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Pin}^-(2))/\{\pm 1\} \ni [q_+, q_-, u]$.

$\mathrm{Spin}^{c-}(4)$-modules $\mathbb{H}_T$, $\mathbb{H}_+$ and $\mathbb{H}_-$

- $\mathbb{H}_T$, $\mathbb{H}_+$, $\mathbb{H}_- \cong \mathbb{H}$ as vector spaces.
- The actions of $[q_+, q_-, u] \in \mathrm{Spin}^{c-}(4)$ are given by

$$\mathbb{H}_T \ni v \mapsto q_+ v q_-^{-1} \quad \longrightarrow P \times_{\mathrm{Spin}^{c-}(4)} \mathbb{H}_T \cong TX$$
$$\mathbb{H}_\pm \ni \phi \mapsto q_\pm \phi u^{-1} \quad \longrightarrow P \times_{\mathrm{Spin}^{c-}(4)} \mathbb{H}_\pm =: S^\pm$$

$S^\pm$ are the positive/negative spinor bundles.

The Clifford multiplication

Define the $\mathrm{Spin}^{c-}(4)$-equivariant map

$$\rho_0 \colon \mathbb{H}_T \times \mathbb{H}_+ \to \mathbb{H}_-, (v, \phi) \mapsto \bar{v}\phi.$$
$$\longrightarrow \rho \colon \Omega^1(X) \times \Gamma(S^+) \to \Gamma(S^-).$$

Twisted complex version

- $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$ has two components.
- Let $G_0 \subset \mathrm{Spin}^{c-}(4)$ be the identity component.
- Let $\varepsilon \colon \mathrm{Spin}^{c-}(4) \to \mathrm{Spin}^{c-}(4)/G_0 \cong \{\pm 1\}$ be the projection.
  $\longrightarrow P \times_\varepsilon \mathbb{R} = \det E =: \lambda$
- Let $\mathrm{Spin}^{c-}(4)$ act on $\mathbb{C}$ by complex conjugation via ε.
- Define the $\mathrm{Spin}^{c-}(4)$-equivariant map,

$$\rho_0 \colon \mathbb{H}_T \otimes_{\mathbb{R}} \mathbb{C} \times \mathbb{H}_+ \to \mathbb{H}_-, (v \otimes a, \phi) \mapsto \bar{v}\phi\bar{a}.$$
$$\longrightarrow \rho \colon \Omega^1(\underline{\mathbb{R}} \oplus i\lambda) \times \Gamma(S^+) \to \Gamma(S^-).$$

Dirac operator

An $\mathrm{O}(2)$-connection A on E + Levi-Civita connection
$\to$ A $\mathrm{Spin}^{c-}(4)$-connection $\mathbb{A}$ on P
$\to$ Dirac operator

$$D_A \colon \Gamma(S^+) \to \Gamma(S^-).$$

If A' is another $\mathrm{O}(2)$-connection $\Rightarrow a = A - A' \in \Omega^1(i\lambda)$.

$$D_{A+a}\phi = D_A\phi + \rho(a)\phi.$$

Quadratic map

Let $x = [q_+, q_-, u] \in \mathrm{Spin}^{c-}(4)$ act on $\mathrm{im}\,\mathbb{H}$ by

$$\mathrm{im}\,\mathbb{H} \ni v \mapsto \varepsilon(x) q_+ v q_+^{-1} \quad \longrightarrow \Gamma(P \times_{\mathrm{Spin}^{c-}(4)} \mathrm{im}\,\mathbb{H}) \cong \Omega^+(i\lambda).$$

Then $\phi \in \mathbb{H}_+ \mapsto \phi i \bar{\phi} \in \mathrm{im}\,\mathbb{H}$ is $\mathrm{Spin}^{c-}(4)$-equivariant. We obtain

$$q \colon \Gamma(S^+) \to \Omega^+(i\lambda).$$

$\mathrm{Pin}^-(2)$-monopole equations

Let $\mathcal{A}$ be the space of $\mathrm{O}(2)$-connections on E.
For $(A, \phi) \in \mathcal{A} \times \Gamma(S^+)$, $\mathrm{Pin}^-(2)$-monopole equations are defined by

$$\begin{cases} D_A\phi = 0, \\ F_A^+ = q(\phi). \end{cases}$$

Relation to Seiberg-Witten theory

- $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$ has two component.
- The identity compo. $G_0 = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1) = \mathrm{Spin}^c(4)$.
- $\mathrm{Spin}^{c-}(4)/G_0 = \mathbb{Z}/2$.

- Let (P, τ) be a Spin^{c-}-structure on (X, E).
- $\tilde{X} = P/G_0 \to X$ is a double covering s.t.

$$\lambda := \tilde{X} \times_{\{\pm 1\}} \mathbb{R} \cong \det E.$$

- $P \to \tilde{X}$ is a $G_0 = \mathrm{Spin}^c(4)$-bundle.

$$\begin{array}{ccc}
P & = & P \curvearrowleft J \\
\downarrow {\scriptstyle \mathrm{Spin}^{c-}(4)} & & \downarrow {\scriptstyle G_0 = \mathrm{Spin}^c(4)} \\
X & \xleftarrow[2:1]{} & P/G_0 = \tilde{X} \curvearrowleft \iota
\end{array}$$

- $\iota\colon \tilde{X} \to \tilde{X}$, the covering transformation.
- $J = [1, 1, j] \in (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Pin}^-(2))/\{\pm 1\} = \mathrm{Spin}^{c-}(4)$
- The Spin^c-structure c on $\tilde{X}$ is induced from $P \to \tilde{X}$.
- The J-action induces antilinear involutions I on the spinor bundles and the determinant line bundle of c.

$\mathrm{Pin}^-(2)$-monopole theory on X = I-invariant SW theory on $\tilde{X}$.

Gauge transformation group

$$\begin{aligned}\mathcal{G} :=&\{\mathrm{Spin}^{c-}(4)\text{-equiv. diffeos of } P \text{ covering the id. of } P/\mathrm{Pin}^-(2)\}\\ \cong&\Gamma(P\times_{\mathrm{ad}}\mathrm{Pin}^-(2)),\end{aligned}$$

where "ad" is the adjoint action on $\mathrm{Pin}^-(2)$ by $\mathrm{Pin}^-(2)$-compo. of $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4)\times_{\{\pm1\}}\mathrm{Pin}^-(2)$.

$g\in\mathcal{G}$ acts on $(A,\phi)\in\mathcal{A}\times\Gamma(S^+)$ by $g(A,\phi) = (A-2g^{-1}dg, g\phi)$.

Cf. In the SW-case, $\mathcal{G}_{SW} = \mathrm{Map}(X, S^1)$.

The moduli space $\mathcal{M} = \{$ solutions $\}/\mathcal{G}$.

What is $\mathcal{G} = \Gamma(P\times_{\mathrm{ad}}\mathrm{Pin}^-(2))$?

- $\mathrm{Pin}^-(2) = \mathrm{U}(1)\cup j\,\mathrm{U}(1)$.

For $u, z\in\mathrm{U}(1)$,
$$\begin{aligned}\mathrm{ad}_z(u) &= zu\bar z = u,\\ \mathrm{ad}_{jz}(u) &= jzu\bar z(-j) = \bar u,\\ \mathrm{ad}_z(ju) &= z^2 ju,\\ \mathrm{ad}_{jz}(ju) &= \bar z^2 j\bar u.\end{aligned}$$
$$\Rightarrow \mathcal{G} = \mathcal{G}_0\cup\mathcal{G}_1,\qquad \begin{aligned}\mathcal{G}_0 &= \Gamma(P\times_{\mathrm{ad}}\mathrm{U}(1)),\\ \mathcal{G}_1 &= \Gamma(P\times_{\mathrm{ad}} j\,\mathrm{U}(1)).\end{aligned}$$

- Note $\mathcal{G}_0\cong\Gamma(\tilde X\times_{\{\pm1\}}\mathrm{U}(1))$, where $\{\pm1\}$ acts on $\mathrm{U}(1)$ by complex conjugation.

Define the involution I on $\mathcal{G}_{SW} = \mathrm{Map}(\tilde X; S^1)$ by $Ig = \overline{\iota^* g}$, where $\iota\colon\tilde X\to\tilde X$ the covering transformation. $\Rightarrow$ $\boxed{\mathcal{G}_0 = (\mathcal{G}_{SW})^I.}$

Proposition $\mathcal{G}_1 = \Gamma(P \times_{\mathrm{ad}} j\,\mathrm{U}(1)) \neq \emptyset \Leftrightarrow \tilde{c}_1(E) = 0.$

- $\tilde{c}_1(E)$ is the Euler class considered in $H^2(X;l)$, where l is the sub-$\mathbb{Z}$-bundle of $\lambda = \det E$.
  Froyshov calls $\tilde{c}_1(E)$ the *twisted 1st Chern class.*
- The iso. classes of $\mathrm{O}(2)$-bundle E over X s.t. $\det E \cong \lambda$ are classified by $\tilde{c}_1(E) \in H^2(X;l)$. $\leftarrow$ Proved by Froyshov.
- $\tilde{c}_1(E) = 0 \Leftrightarrow E \cong \underline{\mathbb{R}} \oplus \lambda$.
- Since $\mathrm{ad}_z(ju) = z^2 ju$ & $\mathrm{ad}_{jz}(ju) = \bar{z}^2 j\bar{u}$,

$$P \times_{\mathrm{ad}} j\,\mathrm{U}(1) \cong S(E): \text{ The bundle of unit vectors of } E.$$

The moduli space

$$\mathcal{M} = \{ \text{ solutions } \}/\mathcal{G},$$
$$\mathcal{M}_0 = \{ \text{ solutions } \}/\mathcal{G}_0.$$

Note $\tilde{c}_1(E) \neq 0 \Rightarrow \mathcal{G} = \mathcal{G}_0 \Rightarrow \mathcal{M} = \mathcal{M}_0$.

Proposition

- $\mathcal{M}$ is compact.
- The virtual dimension of $\mathcal{M}$:

$$d = \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b_0(X;l) - b_1(X;l) + b_+(X;l)).$$

If l is nontrivial & X connected $\Rightarrow b_0(X;l) = 0$.

Reducibles

- Recall $g(A,\phi) = (A - 2g^{-1}dg, g\phi)$.
- If $\phi \neq 0 \Rightarrow \mathcal{G}$-action is free.
- The stabilizer of $(A,0)$ is $\{\pm 1\} \subset \mathcal{G}_0 \cong \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1))$, unless $E = \underline{\mathbb{R}} \oplus \lambda$ and A is flat ($\Rightarrow$ The stabilizer $\cong \mathbb{Z}/4$).
- The elements of the form $(A,0)$ are called reducibles.

Cf. In the SW-case, the stabilizer of $(A,0)$ is $S^1 \subset \mathrm{Map}(X,S^1)$.

- In general, $\{$ reducible solutions $\}/\mathcal{G}_0 \cong T^{b_1(X;l)} \subset \mathcal{M}_0$.

Proof of Theorem 1

Theorem 1.(N.)

- X: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

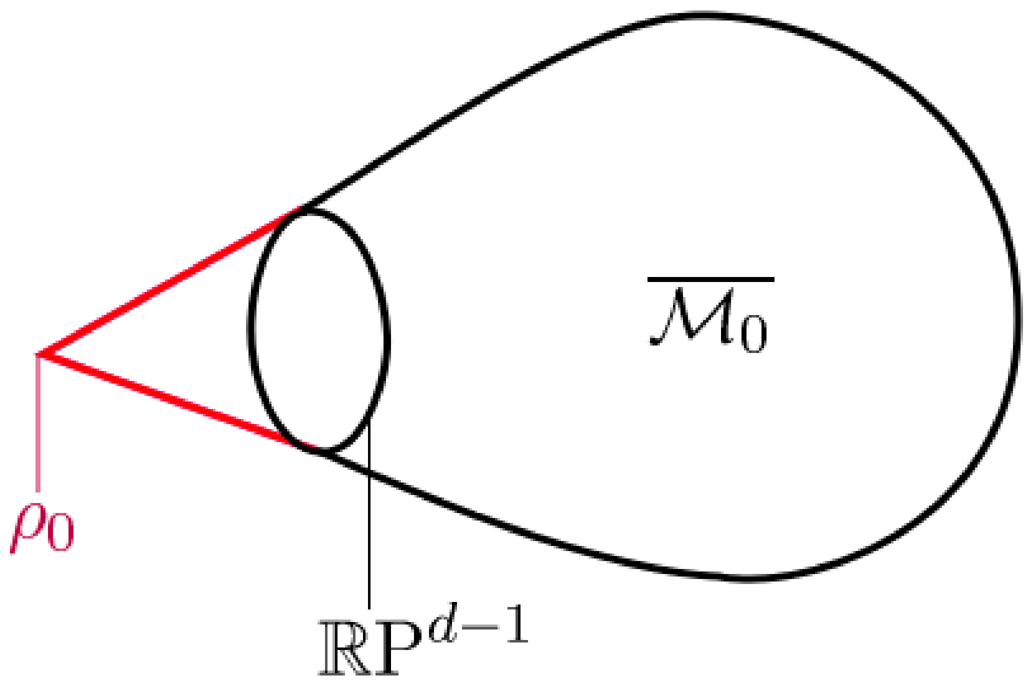
Outline of the proof

- We will prove every characteristic element w of $Q_{X,l}$ satisfies
$$|w^2| \geq \mathrm{rank}\, H^2(X;l),$$
by proving for every E,
$$d = \dim \mathcal{M}_0 \leq 0.$$
- Then Elkies' theorem implies $Q_{X,l}$ should be standard.

The structure of $\mathcal{M}_0$ when $b_+(X;l)=0$

- Suppose a Spin^{c-}-structure (P,τ) on X is given.
- For simplicity, assume $b_1(X,l)=0$.
  $\Rightarrow \exists^1$ reducible class $\rho_0 \in \mathcal{M}_0$.
- Perturb the $\mathrm{Pin}^{-}(2)$-monopole equations by adding $\eta \in \Omega^+(i\lambda)$ to the curvature equation. $\to F_A^+ = q(\phi)+\eta$.
- For generic η, $\mathcal{M}_0 \setminus \{\rho_0\}$ is a d-dimensional manifold.
- Fix a small neighborhood $N(\rho_0)$ of $\{\rho_0\}$.
  $\Rightarrow N(\rho_0) \cong \mathbb{R}^d/\{\pm 1\} =$ a cone of $\mathbb{RP}^{d-1}$

Then $\overline{\mathcal{M}_0} := \overline{\mathcal{M}_0 \setminus N(\rho_0)}$ is a compact d-manifold & $\partial\overline{\mathcal{M}_0} = \mathbb{RP}^{d-1}$.



- Let $\mathcal{B}^* = (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\})) / \mathcal{G}_0$.

Proposition $\mathcal{B}^* \underset{h.e.}{\simeq} \mathbb{R}\mathrm{P}^\infty \times T^{b_1(X;l)}$.

Cf. In the SW-case, $\mathcal{B}^*_{SW} \underset{h.e.}{\simeq} \mathbb{C}\mathrm{P}^\infty \times T^{b_1(X)}$. $\mathcal{B}^* \cong (\mathcal{B}^*_{SW})^I$.

Lemma

If $b_+(X;l) = 0$ & $b_1(X;l) = 0 \Rightarrow d = \dim \mathcal{M}_0 \leq 0$.

Proof

- Suppose $d > 0$.
- Recall $\overline{\mathcal{M}_0}$ is a compact d-manifold s.t. $\partial\overline{\mathcal{M}_0} = \mathbb{R}\mathrm{P}^{d-1}$.
- $\exists C \in H^{d-1}(\mathcal{B}^*; \mathbb{Z}/2) \cong H^{d-1}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$ s.t. $\langle C, [\partial\overline{\mathcal{M}_0}]\rangle \neq 0$. $\Rightarrow$ Contradiction.

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Proof of Theorem 1 & 2
Proof of Theorem 1
Proof of Theorem 2

- Note $\mathrm{sign}(X) = b_+(X;l) - b_-(X;l)$ for any $\mathbb{Z}$-bundle l.
- By Lemma, if l is nontrivial & $b_+(X;l) = 0$ & $b_1(X;l) = 0$,

$$\begin{aligned} d =& \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b_0(X;l) - b_1(X;l) + b_+(X;l)) \\ =& \frac{1}{4}(\tilde{c}_1(E)^2 + b_2(X;l)) \leq 0. \end{aligned}$$

Note $\tilde{c}_1(E)^2 \leq 0$ if $b_+(X;l) = 0$.

- Therefore, for any E which admits a Spin^{c-}-structure,

$$b_2(X;l) \leq |\tilde{c}_1(E)^2|.$$

By varying E, we can prove every characteristic element w satisfies

$$b_2(X;l) \leq |w^2|.$$

Recall

- E admits a Spin^{c-}-structure
  $\Leftrightarrow w_2(X) = w_2(E) + w_1(E)^2 = w_2(E) + w_1(\lambda)^2$,
  where $\lambda = \det E = l \otimes \mathbb{R}$.
- $\tilde{c}_1(E) \in H^2(X; l)$ classifies E s.t. $\det E = l \otimes \mathbb{R}$.

Note that $0 \to l \xrightarrow{\cdot 2} l \to \mathbb{Z}/2 \to 0$ induces the mod-2-reduction map $[\cdot]_2 \colon H^2(X; l) \to H^2(X; \mathbb{Z}/2)$ & $[\tilde{c}_1(E)]_2 = w_2(E)$. We have,

Theorem

Suppose $w_1(\lambda)^2 = 0$. For every $C \in H^2(X; l)$ s.t. $w_2(X) = [C]_2 + w_1(\lambda)^2 = [C]_2$,

$$|C^2| \geq b_2(X; l).$$

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Lemma

For every characteristic element c of $Q_{X,l}$, $\exists$ a torsion $\delta \in H^2(X; l)$ s.t. $[c + \delta]_2 = w_2(X)$.

Then, for $\forall$ characteristic element c of $Q_{X,l}$

$$|c^2| = |(c + \delta)^2| \geq b_2(X; l).$$

By Elkies' theorem, $Q_{X,l}$ $\sim$diagonal.

Nobuhiro Nakamura $\mathrm{Pin}^-(2)$-monopole equations and intersection forms

The outline of the proof of Theorem 2

- Suppose $w_1(\lambda)^2 = w_2(X)$. Let $E = \underline{\mathbb{R}} \oplus \lambda$.
  $\Rightarrow \exists\,\mathrm{Spin}^{c-}$-structure on (X, E). $\Rightarrow \mathcal{G}_1 \neq \emptyset$.
- For simplicity, assume $b_1(X; l) = 0$.
- Then, by taking finite dimensional approximation of the monopole map, we obtain a proper $\mathbb{Z}_4$-equivariant map

$$f\colon \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

  where
  - $\tilde{\mathbb{R}}$ is $\mathbb{R}$ on which $\mathbb{Z}_4$ acts via $\mathbb{Z}_4 \to \mathbb{Z}_2 = \{\pm 1\} \curvearrowright \mathbb{R}$,
  - $\mathbb{C}_1$ is $\mathbb{C}$ on which $\mathbb{Z}_4$ acts by multiplication of i,
  - $k = -\operatorname{sign}(X)/8$, $b = b_+(X;\lambda)$, m, n are some integers.

  Here, $\mathbb{Z}_4$ is generated by the constant section

$$j \in \mathcal{G}_1 = \Gamma(\tilde{X} \times_{\{\pm 1\}} j\,\mathrm{U}(1)).$$

- By using the techniques of equivariant homotopy theory, e.g., tom Dieck's character formula, we can see that any proper $\mathbb{Z}_4$-map of the form,

$$f\colon \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

  should satisfy $b \geq k$.
- That is,

$$b_+(X;\lambda) \geq -\frac{1}{8}\operatorname{sign}(X).$$

Finite dimensional approximation

- Take a flat connection A_0 on $\underline{\mathbb{R}} \oplus \lambda$.

$\mathrm{Pin}^-(2)$-monopole map

$$\mu\colon \Omega^1(i\lambda) \oplus \Gamma(S^+) \to (\Omega^0 \oplus \Omega^+)(i\lambda) \oplus \Gamma(S^-) =: \mathcal{W},$$
$$(a, \phi) \mapsto (d^*a, F_{A_0} + d^+a + q(\phi), D_{A_0+a}\phi).$$

- Let $l(a, \phi) := (d^*a, d^+a, D_{A_0}\phi)$ be the linear part of μ. $\rightarrow$ l is Fredholm.
- $c = \mu - l$: quadratic, compact.
- Choose a finite dim. subspace $U \subset \mathcal{W}$ s.t. $\dim U \gg 1$, $U \supset (\operatorname{im} l)^\perp$
- Let $V := l^{-1}(U)$ & $p\colon \mathcal{W} \to U$ be the L^2-projection.
- Define $f\colon V \to U$ by $\boxed{f = l + pc.}$ $\rightarrow$ f: proper, $\mathbb{Z}_4$-equiv.

Introduction
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$\mathrm{Pin}^-(2)$-monopole equations
Proof of Theorem 1 & 2
Proof of Theorem 1
Proof of Theorem 2

Remarks for future researches

- $\mathrm{Pin}^-(2)$-monopole invariants
  - Calculation, gluing formula, stable cohomotopy refinements
- Orbifolds with surface singularities
  - Exotic involutions *Cf.* [Fintushel-Stern-Snukujian]
  - Smooth inequivalent but topologically equivalent embedded surfaces *Cf.* [H.J.Kim-Ruberman]
- When $\tilde{X}$: symplectic & $I^*\omega = -\omega$,

  $$\mathrm{Pin}^-(2)\text{-monopole inv.} \underset{??}{=} \text{real Gromov-Witten inv.}$$

  Cf. [Tian-Wang]
- $\mathrm{Pin}^-(2)$-monopole Floer theory?
  $\mathrm{Pin}^-(2)$ Heegaard Floer theory?
- "Witten conjecture" for $\mathrm{Pin}^-(2)$-monopole invariants?
  - [Feehan-Leness] SW $=$ Donaldson

    $\mathrm{Pin}^-(2)$-monopole inv. $= ???$

Nobuhiro Nakamura $\mathrm{Pin}^-(2)$-monopole equations and intersection forms