# NONSMOOTHABLE GROUP ACTIONS IN DIMENSION 4

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Abstract. In this note, we survey our recent works on nonsmoothable locally linear group actions on 4-manifolds. The first part deals with joint works with X. Liu on nonsmoothable $\mathbb{Z}_p$-actions on elliptic surfaces. The second part deals with a joint work with Y. Fukumoto on nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds whose boundaries are Brieskorn homology 3-spheres.

## 1. Introduction

It is a classical result that every finite group action on a surface is equivalent to a smooth one. In higher dimensions, there exist examples of nonsmoothable actions. Since bad local behavior is often the reason why these actions can not be smooth, one can naturally ask whether *locally linear* actions are smoothable or not. In [24], S. Kwasik and K. B. Lee proved that in dimension 3 a finite group action is smoothable if and only if it is locally linear. However, in dimensions higher than 3, this is not true. In fact, many examples of nonsmoothable locally linear actions are known [24, 23, 20, 6, 22, 8].

In this note, we survey our recent works on nonsmoothable locally linear group actions on 4-manifolds. The first part deals with joint works with X. Liu on nonsmoothable $\mathbb{Z}_p$-actions on elliptic surfaces [25, 26]. The second part deals with a joint work with Y. Fukumoto on nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds whose boundaries are Brieskorn 3-spheres [16].

In [25, 26], X. Liu and the author constructed a lot of examples of nonsmoothable cyclic group actions on elliptic surfaces.

**Theorem 1.1** ([25, 26]). *Let $\mathbb{Z}_p$ be the cyclic group of order $p = 3$, 5 or 7. For each even integer $n \geq 2$, let $X_n = E(n)$ be the simply-connected relatively minimal elliptic surface over $S^2$ whose Euler number $e(X_n)$ is $12n$. Let*

$$c_{n-2} := \begin{pmatrix} n-2 \\ \dfrac{n-2}{2} \end{pmatrix}. \tag{1.2}$$

*(We assume $c_0 = 1$.) If $c_{n-2} \not\equiv 0 \bmod p$, then there exists a locally linear $\mathbb{Z}_p$-action on $X_n$ which is nonsmoothable with respect to infinitely many smooth structures on $X_n$ including the standard one.*

*Remark* 1.3. We do not know whether there exists a smooth structure on $X_n$ on which the above action is smoothable, or not.

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*Remark* 1.4. The number $c_{n-2}$ is the Seiberg-Witten invariant of $E(n)$ for the spin structure.

In general, the proof of the existence of nonsmoothable locally linear actions is divided into two steps: (1) to give a constraint on smooth actions, (2) to construct a locally linear action which would violate the constraint.

In our case, to construct locally linear actions, we invoke a remarkable realization theorem by Edmonds and Ewing [10]. On the other hand, to give a constraint on smooth actions, we use the Seiberg-Witten gauge theory. In fact, we use a mod $p$ vanishing theorem of Seiberg-Witten invariants under $\mathbb{Z}_p$-actions, and the calculation of the invariants for elliptic surfaces.

The mod $p$ vanishing theorem (Theorem 2.12), which is originally proved by F. Fang [11], and generalized by the author [28], claims that, if some conditions on fixed point data are satisfied, then the Seiberg-Witten invariant is divisible by $p$. Therefore, if the Seiberg-Witten invariant is not divisible by $p$, then a constraint on fixed point data is obtained. Obviously, this method can not be used when the Seiberg-Witten invariant is 0.

On the other hand, S. Bauer and M. Furuta defined a stable cohomotopy version of the Seiberg-Witten invariants [5], and their invariants are true refinement of the original Seiberg-Witten invariants. For example, all the Seiberg-Witten invariants of $K3\#K3$ are 0, but the Bauer-Furuta invariant of it for the spin structure is non-zero [4, 17].

In [29], the author proved a vanishing theorem for Bauer-Furuta invariants under $\mathbb{Z}_2$-actions, which has a similar form to the aforementioned mod $p$ vanishing theorem, and constructed a locally linear involution on $K3\#K3$ by using the vanishing theorem.

**Theorem 1.5** ([29]). *There exists a locally linear $\mathbb{Z}_2$-action on $X = K3\#K3$ which is nonsmoothable with respect to any smooth structure on $X$.*

As a byproduct of our argument, we also have a nonsmoothable involution on $K3$.

**Theorem 1.6** ([29]). *There exists a locally linear $\mathbb{Z}_2$-action on $K3$ with isolated fixed points satisfying $b_+^{\mathbb{Z}_2} = \dim H^+(X;\mathbb{R})^{\mathbb{Z}_2} = 3$ which is nonsmoothable with respect to any smooth structure on $K3$.*

It would be interesting to compare this with a result by J. Bryan:

**Theorem 1.7** (Bryan [6]). *Every smooth $\mathbb{Z}_2$-action on $K3$ with isolated fixed points satisfies $b_+^{\mathbb{Z}_2} = 3$*

Recently, Y. Fukumoto and the author began to study nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds whose boundaries are Brieskorn homology 3-spheres by the Seiberg-Witten gauge theory [16]. Such nonsmoothable actions are studied by Kwasik-Lawson in [23] very widely and systematically, and they used the invariant of Fintushel-Stern [12] in order to prove the nonsmoothability.

We are interested in the following questions:

(1) Can we prove Kwasik-Lawson's results by using the Seiberg-Witten gauge theory, especially by Fukumoto-Furuta's $w$-invariants [15], instead of Fintushel-Stern's invariants?

(2) Is there difference between consequences from $w$-invariants and Fintushel-Stern's invariants?

At present, we have a partial result for (1). That is, for several examples, Kwasik-Lawson's results can be obtained by using $w$-invariants. To state our results, we need some preliminaries.

Let $\Sigma(a, b, c)$ be the Brieskorn variety. If $a$, $b$ and $c$ are mutually coprime, then $\Sigma(a, b, c)$ is an integral homology 3-sphere. For an integer $p$, there is the standard $\mathbb{Z}_p$-action on $\Sigma(a, b, c)$ which is a part of the circle action of the Seifert fibration.

Kwasik-Lawson [23] proved the following result on the extension of the standard action to a *locally linear* action on a contractible 4-manifold $W$ whose boundary is $\Sigma(a, b, c)$:

**Theorem 1.8** ([23], *cf.* [9]). *Let $p$ be an odd prime. There exist a contractible 4-manifold $W$ whose boundary is $\Sigma(a, b, c)$, and a locally linear $\mathbb{Z}_p$-action on $W$ which extends the standard action on $\Sigma(a, b, c)$ and has exactly one fixed point of type $(r, s)$ if and only if $\{a, b, c\}$ is congruent modulo $p$ to $\{r, s, 1\}$ up to sign, and $abc \equiv rs \bmod p$.*

*Remark* 1.9. This theorem is based on [9], and is a part of the proof of Edmonds-Ewing's theorem (Theorem 2.8). See Remark 2.11.

Note that Casson-Harer [7] and several other authors proved that there are infinitely many families of $\Sigma(a, b, c)$ which can be boundaries of *smooth* contractible 4-manifolds $W$. With these understood, our result is;

**Theorem 1.10** ([16]). *For $(a, b, c)$ and $p$ below, $\Sigma(a, b, c)$ bounds a smooth contractible 4-manifold $W$, and the standard $\mathbb{Z}_p$-action extends locally linearly over $W$, but there is no such smooth action.*

(1) $p = 3$, $(a, b, c) = (2, 11, 53)$, $(5, 13, 14)$, $(5, 16, 17)$, $(5, 31, 32)$, $(5, 46, 47)$, $(11, 31, 32)$.
(2) $p = 5$, $(a, b, c) = (11, 31, 32)$. $(3, 13, 14)$, $(3, 16, 17)$, $(3, 28, 29)$, $(3, 31, 32)$, $(3, 43, 44)$.
(3) $p = 7$, $(a, b, c) = (3, 19, 20)$, $(3, 22, 23)$, $(3, 40, 41)$, $(5, 33, 34)$, $(5, 36, 37)$, $(5, 68, 69)$.

The organization of this note is as follows: Section 2 gives some preliminaries. Section 3 proves a part of Theorem 1.1 in the case when $G = \mathbb{Z}_3$ and $X = K3$. In Section 4, the proof of Theorem 1.10 is explained.

## 2. Preliminaries

In this section, we give some preliminaries, and collect several known results on locally linear and smooth group actions.

2(i). **Locally linear actions.** First, let us recall the definition of locally linear actions of finite groups.

**Definition 2.1.** Let $G$ be a finite group, and $X$ a topological $n$-manifold. A topological $G$-action on $X$ is called *locally linear* if, for any $x \in X$, there exists a neighborhood $V_x$ which is invariant under the action of the isotropy $G_x$ of $x$ which satisfies,

(1) $V_x$ is homeomorphic to $\mathbb{R}^n$,
(2) $G_x$ acts on $V_x \cong \mathbb{R}^n$ in a linear orthogonal way.

*Remark* 2.2. In general, a smooth action is locally linear. However, the converse is not true.

Let us recall what is nonsmoothable group actions. Let $X$ be a *topological* manifold and $G$ a finite group. If $X$ admits a *smooth* structure and a smooth structure $\sigma$ is specified, then we write the manifold with the smooth structure $\sigma$ by $X_\sigma$. Let $LL(G, X)$ be the set of equivalence classes of locally linear $G$-actions on $X$. Here, two locally linear actions are said equivalent if there exists a *homeomorphism* $f$ of $X$ such that one action is conjugate to the other by $f$. Similarly, let $C^\infty(G, X_\sigma)$ be the set of equivalence classes of smooth $G$-actions on $X_\sigma$. Here, the equivalence of two smooth actions is given via a *diffeomorphism* of $X_\sigma$. Then we have a forgetful map $\Phi_\sigma\colon C^\infty(G, X_\sigma) \to LL(G, X)$ forgetting the smooth structure.

**Definition 2.3.** A locally linear action is called *nonsmoothable* with respect to the smooth structure $\sigma$ if its class is not contained in the image of $\Phi_\sigma$.

2(ii). **The $G$-index theorems.** For the generality of $G$-index theorems, we refer [1, 2, 3, 31]. Let $G = \mathbb{Z}_p$ be a cyclic group of odd prime order $p$. Suppose a smooth $G$-action on a closed smooth 4-manifold $X$ is given. Suppose further that the $G$-action is *pseudofree*, i.e., $G$ acts freely on the complement of a discrete subset. When we fix a generator $g$ of $G = \mathbb{Z}_p$, the representation at a fixed point can be described by a pair of nonzero integers $(a, b)$ modulo $p$ which is well-defined up to order and changing the sign of both together. Suppose that the fixed point data for the generator $g$ are given as $\{(a_i, b_i)\}_{i=1}^N$.

Then the *$G$-signature formula* is

$$\operatorname{Sign}(g, X) = \sum_{i=1}^{N} s_{a_i b_i} \tag{2.4}$$

where

$$s_{xy} = \frac{(\zeta^x + 1)(\zeta^y + 1)}{(\zeta^x - 1)(\zeta^y - 1)}, \tag{2.5}$$

and $\zeta = \exp(2\pi\sqrt{-1}/p)$.

Suppose further that $X$ is *spin* and the $G$-action is a *spin* action. Let $D_X$ be the $G$-equivariant Dirac operator. Then the *$G$-spin theorem* is

$$\operatorname{ind}_g D_X = \sum_{i=1}^{N} p_{a_i b_i}, \tag{2.6}$$

where

$$p_{xy} = \frac{1}{(\zeta^x)^{1/2} - (\zeta^x)^{-1/2}} \frac{1}{(\zeta^y)^{1/2} - (\zeta^y)^{-1/2}}, \tag{2.7}$$

and signs of $(\zeta^x)^{1/2}$ and $(\zeta^y)^{1/2}$ are determined by the rule

$$\left\{(\zeta^x)^{1/2}\right\}^p = \left\{(\zeta^y)^{1/2}\right\}^p = 1.$$

(This is because $G$ is supposed odd order, and the $g$-action generates the $G$-action on the Spin-structure. See [3, p.20] or [31, p.175].)

2(iii). **The realization theorem by Edmonds and Ewing.** We summarize the realization theorem of locally linear pseudofree actions by A. L. Edmonds and J. H. Ewing [10] in the very special case when $G = \mathbb{Z}_3, \mathbb{Z}_5$ or $\mathbb{Z}_7$.

**Theorem 2.8** ([10]). *Let $G$ be the cyclic group of order $p$, where $p = 3, 5$ or $7$. Suppose that one is given a fixed point data*

$$\mathcal{D} = \{(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n), (a_{n+1}, b_{n+1})\},$$

*where $a_i, b_i \in \mathbb{Z}_p \setminus \{0\}$, and a $G$-invariant symmetric unimodular form*

$$\Phi\colon V \times V \to \mathbb{Z},$$

*where $V$ is a finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module. Then the data $\mathcal{D}$ and the form $(V, \Phi)$ are realizable by a locally linear, pseudofree, $G$-action on a closed, simply-connected, topological 4-manifold if and only if they satisfy the following two conditions*:

1. *The condition REP*: *As a $\mathbb{Z}[G]$-module, $V$ splits into $F \oplus T$, where $F$ is free and $T$ is a trivial $\mathbb{Z}[G]$-module with $\operatorname{rank}_{\mathbb{Z}} T = n$.*
2. *The condition GSF*: *The $G$-Signature Formula is satisfied*:

$$\operatorname{Sign}(g, (V, \Phi)) = \sum_{i=0}^{n+1} \frac{(\zeta^{a_i} + 1)(\zeta^{b_i} + 1)}{(\zeta^{a_i} - 1)(\zeta^{b_i} - 1)}, \tag{2.9}$$

*where $\zeta = \exp(2\pi\sqrt{-1}/p)$.*

*Remark* 2.10. In [10], A. L. Edmonds and J. H. Ewing prove the realization theorem for all cyclic groups of prime order $p$, and for general $p$, the third condition *TOR* which is related to the Reidemeister torsion should be satisfied. However, when $p$ is a prime less than 23, the condition *TOR* is redundant. This follows from the fact that the class number of $\mathbb{Z}[\zeta]$ is 1, and Corollary 3.2 of [10].

*Remark* 2.11. The proof of the "if" direction of Theorem 2.8 is given by an equivariant handle construction: By attaching $G$-equivariant 0- and 2-handles according to the given algebraic data, we obtain a 4-manifold $X'$ with a smooth $G$-action whose boundary is an integral homology 3-sphere with a free $G$-action. Then, the required manifold $X$ with a locally linear action is obtained from gluing $X'$ with a contractible 4-manifold $W$ with a locally linear $G$-action as in Theorem 1.8. Note that the resulting locally linear $G$-action is smooth except one point.

2(iv). **Mod $p$ vanishing theorem of Seiberg-Witten invariants.** Let $p$ be a prime, and suppose that $G = \mathbb{Z}_p$ acts on a smooth 4-manifold $X$ *smoothly*. Fix a $G$-invariant metric. Suppose that the $G$-action lifts to a Spin$^c$-structure $c$. Fix a $G$-invariant connection $A_0$ on the determinant line bundle $L$ of $c$. Then the Dirac operator $D_{A_0}$ associated to $A_0$ is $G$-equivariant, and the $G$-index of $D_{A_0}$ can be written as $\operatorname{ind}_G D_{A_0} = \sum_{j=0}^{p-1} k_j \mathbb{C}_j \in R(G) \cong$

$\mathbb{Z}[t]/(t^p-1)$, where $\mathbb{C}_j$ is the complex 1-dimensional weight $j$ representation of $G$ and $R(G)$ is the representation ring of $G$.

In such a situation, the following theorem is proved.

**Theorem 2.12** ([11, 28]). *Let $G$ be the cyclic group of prime order $p$, and $X$ be a smooth closed oriented 4-dimensional $G$-manifold with $b_1 = 0$, $b_+ \geq 2$ and $b_+^G \geq 1$. Suppose that the $G$-action lifts to a* $\mathrm{Spin}^c$*-structure $c$. If $2k_j < 1 + b_+^G$ for $j = 0, 1, \dots, p-1$, then the Seiberg-Witten invariant* $\mathrm{SW}_X(c)$ *for $c$ satisfies*

$$\mathrm{SW}_X(c) \equiv 0 \mod p.$$

*Remark* 2.13. In [11], Fang suppose that $b_+^G = b_+$. In [28], the author weakened that condition as above, and generalized to the case when $b_1 \geq 1$.

*Remark* 2.14. Suppose $X$ is spin and has no 2-torsion in $H_1(X;\mathbb{Z})$. Let $c_0$ be the $\mathrm{Spin}^c$-structure which is determined by a Spin-structure, whose determinant line bundle $L$ is trivial. If $p$ is an odd prime, then every $G = \mathbb{Z}_p$-action on $X$ has a spin lift. In other words, there exist a $G$-action on $c_0$ such that the induced $G$-action on $L$ is the product of the $G$-action on $X$ and the trivial $G$-action on fiber. (See e.g., Lemma 5.7 in [28].) Let $A_0$ be the trivial flat $G$-invariant connection on $L$. Then $\mathrm{ind}_g D_{A_0}$ can be calculated by the $G$-spin theorem (2.6).

## 3. Proof of Theorem 1.1 in the case of $\mathbb{Z}_3$-actions on $K3$

In this section, we give a proof of Theorem 1.1 in the simplest case: $G = \mathbb{Z}_3$ and $X = K3$. The argument in this section is a prototype of the proof of the general case of Theorem 1.1.

3(i). **Possible fixed point data.** By the Lefschetz formula and the $G$-signature formula, we can make the list of candidates for fixed point data of locally linear actions.

In the case when $G = \mathbb{Z}_3$, there are two types of fixed points.

- The type $(+)$: $(1,2) = (2,1)$.
- The type $(-)$: $(1,1) = (2,2)$.

Let $m_+$ be the number of fixed points of the type $(+)$, and $m_-$ be the number of fixed points of the type $(-)$.

Let $X$ be $E(n)$ as a topological manifold. Suppose that a locally linear pseudofree $G$-action on $X$ is given. Put $e = \chi(X)$ and $s = \mathrm{Sign}(X)$. First of all, the ordinary Lefschetz formula should hold: $L(g,X) = 2 + \mathrm{tr}(g|_{H^2(X)}) = \#X^G$. Noting that $\#X^G = m_+ + m_-$ and $2 + \mathrm{tr}(g|_{H^2(X)}) \leq e$, we obtain

$$m_+ + m_- \leq e. \tag{3.1}$$

Note that

$$\chi(X/G) = \frac{1}{3}\{e + 2(m_+ + m_-)\}.$$

Since $\chi(X/G)$ is an integer and $e = 12n$, we have

$$m_+ + m_- \equiv 0 \mod 3. \tag{3.2}$$

By Theorem 2.8, the $G$-Signature Formula should hold:

$$\mathrm{Sign}(g,X)=\mathrm{Sign}(g^2,X)=\frac{1}{3}(m_+-m_-),$$
$$\mathrm{Sign}(X/G)=\frac{1}{3}\left\{s+\frac{2}{3}(m_+-m_-)\right\}.$$

Since $\mathrm{Sign}(X/G)$ is an integer,

$$m_+-m_-\equiv -\frac{3}{2}s \mod 9. \tag{3.3}$$

We can calculate $b_+^G$ and $b_-^G$ from $\chi(X/G)$ and $\mathrm{Sign}(X/G)$:

$$b_+^G=\frac{1}{6}\left\{e+s+\frac{1}{3}(8m_++4m_-)\right\}-1, \tag{3.4}$$

$$b_-^G=\frac{1}{6}\left\{e-s+\frac{1}{3}(4m_++8m_-)\right\}-1. \tag{3.5}$$

These should satisfy

$$0\le b_+^G\le b_+,\ 0\le b_-^G\le b_-. \tag{3.6}$$

By (3.1), (3.3), (3.6) and non-negativity of $m_+$ and $m_-$, we obtain the list of the candidates of fixed point data $(m_+,m_-)$. For example, in the case when $X=E(2)=K3$, there are only 4 possible types of fixed point data of locally linear $\mathbb{Z}_3$-actions on $X$ as follows:

Table 1. The types of $\mathbb{Z}_3$-actions on $K3$

| Type | $\#X^G$ | $m_+$ | $m_-$ | $b_2^G$ | $b_+^G$ | $b_-^G$ | $\mathrm{Sign}(X/G)$ |
|---|---|---|---|---|---|---|---|
| $A_0$ | 6 | 6 | 0 | 10 | 3 | 7 | $-4$ |
| $A_1$ | 9 | 3 | 6 | 12 | 3 | 9 | $-6$ |
| $A_2$ | 12 | 0 | 12 | 14 | 3 | 11 | $-8$ |
| $B$ | 3 | 0 | 3 | 8 | 1 | 7 | $-6$ |

3(ii). **A constraint on smooth $\mathbb{Z}_3$-actions on elliptic surfaces.** In this subsection, we give a constraint on smooth $\mathbb{Z}_3$-actions on elliptic surfaces which is obtained from the mod $p$ vanishing theorem (Theorem 2.12).

**Theorem 3.7.** *Let $G=\mathbb{Z}_3$, and $X$ be a closed oriented smooth simply-connected spin 4-manifold which satisfies $2\chi(X)+3\,\mathrm{Sign}(X)=0$. (Note that $X$ is a homotopy $E(n)$.) Suppose $G$ acts on $X$ smoothly and pseudofreely. Let $c_0$ be the* Spin$^c$*-structure determined by the* Spin*-structure, and $\mathrm{SW}_X(c_0)$ be the Seiberg-Witten invariant of $c_0$. If $\mathrm{SW}_X(c_0)\not\equiv 0$ mod 3, then*

$$m_+=0 \text{ or } m_-=0. \tag{3.8}$$

*Proof.* Let $G$ act on $c_0$ such that the $G$-action on the trivial determinant line bundle is the product of the $G$-action on $X$ and the trivial action on fiber as in Remark 2.14. Take the trivial flat connection $A_0$ as the reference $G$-invariant connection. By Theorem 2.12, $\mathrm{SW}_X(c_0) \not\equiv 0 \bmod 3$ implies that there exist $j$ which satisfies $2k_j \geq 1 + b_+^G$.

Note that $b_+^G$ is calculated in (3.4).

Coefficients $k_j$ are calculated by the $G$-spin theorem. By the $G$-spin theorem (2.6), we have

$$\begin{aligned}
\operatorname{ind}_g D_{A_0} &= k_0 + \zeta k_1 + \zeta^2 k_2 = \frac{1}{3}(m_+ - m_-),\\
\operatorname{ind}_{g^2} D_{A_0} &= k_0 + \zeta^2 k_1 + \zeta k_2 = \frac{1}{3}(m_+ - m_-),\\
\operatorname{ind}_1 D_{A_0} &= k_0 + k_1 + k_2 = -\frac{1}{8}s.
\end{aligned}$$

Solving these, we have

$$\begin{aligned}
k_0 &= \frac{2}{9}(m_+ - m_-) - \frac{1}{24}s,\\
k_1 = k_2 &= -\frac{1}{9}(m_+ - m_-) - \frac{1}{24}s.
\end{aligned}$$

¿From these and the relation $2e + 3s = 0$, we have $m_+ = 0$ or $m_- = 0$. □

Now, we recall Seiberg-Witten invariants of elliptic surfaces. Let $E(n)_{k,l}$ be the relatively minimal elliptic surface over $S^2$ with the Euler number $12n$ whose multiple fibers have multiplicities $\{k, l\}$. (We assume $k$, $l$ may be 1.) Suppose that $n$ is even and $\gcd(k, l) = 1$. The last condition implies that $E(n)_{k,l}$ is simply-connected. For such $E(n)_{k,l}$, the following are known:

(1) $E(n)_{k,l}$ is homeomorphic to $\frac{n}{2}K3\#(\frac{n}{2} - 1)S^2 \times S^2$ if and only if $k$ and $l$ are odd and mutually coprime. (See e.g. [32].)
(2) $E(n)_{k,l}$ is diffeomorphic to $E(n)_{k',l'}$ if and only if $\{k, l\} = \{k', l'\}$ as unordered pair. (See [19].)
(3) $E(2)_{1,1} = E(2)$ (no multiple fiber) is diffeomorphic to the standard $K3$ surface.

The Seiberg-Witten invariant of $E(n)_{k,l}$ for the spin structure $c_0$ is given as follows [19]:

$$\mathrm{SW}_{E(n)_{k,l}}(c_0) = (-1)^{\frac{n-2}{2}} \binom{n-2}{\frac{n-2}{2}}. \tag{3.9}$$

Note that this is independent of $k, l$.

By Theorem 3.7, we have,

**Corollary 3.10.** *For every smooth $\mathbb{Z}_3$-action on $E(n)_{k,l}$, if $c_{n-2}$ in (1.2) is not divisible by 3, then $m_+ = 0$ or $m_- = 0$.*

3(iii). **A nonsmoothable $\mathbb{Z}_3$-action on $K3$.** Let $X = K3$ and $G = \mathbb{Z}_3$. Recall Table 1. Since $\mathrm{SW}_X(c_0) = 1$ by (3.9), the data $(m_+, m_-) = (3, 6)$ can not be realized as fixed point data of a smooth action by Corollary 3.10.

Now, we construct a locally linear $G = \mathbb{Z}_3$-action on $X = K3$ with $(m_+, m_-) = (3, 6)$ by Theorem 2.8. The only thing to do is to construct a $G$-invariant form which satisfies the conditions in Theorem 2.8.

Let $(V_{K3}, \Phi_{K3})$ be the intersection form of the $K3$ surface, which is even and indefinite. Since an even indefinite form is completely characterized by its rank and signature, $(V_{K3}, \Phi_{K3})$ is isomorphic to $3H \oplus \Gamma_{16}$, where $H$ is the hyperbolic form, and $\Gamma_{16}$ is a negative definite even form of rank 16 given below. We will construct $G$-actions on $3H$ and $\Gamma_{16}$ separately.

The lattice $\Gamma_{16}$ is the set of $(x_1, \dots, x_{16}) \in (\frac{1}{2}\mathbb{Z})^{16}$ which satisfy

(1) $x_i \equiv x_j \mod \mathbb{Z}$ for any $i, j$,

(2) $\sum_{i=1}^{16} x_i \equiv 0 \mod 2\mathbb{Z}$.

The unimodular bilinear form on $\Gamma_{16}$ is defined by $-\sum_{i=1}^{16} x_i^2$.

**Lemma 3.11.** *For each integer $k$ which satisfies $0 \le k \le 5$, there is a $G$-action on $\Gamma_{16}$ such that*

$$\Gamma_{16} \cong (16 - 3k)\mathbb{Z} \oplus k\mathbb{Z}[G] \text{ as a } \mathbb{Z}[G]\text{-module.}$$

*Proof.* When $k = 0$, it suffices to take the trivial $G$-action. Hence we suppose $k \ge 1$.

Note that the symmetric group of degree 16 acts on $\Gamma_{16}$ as permutations of components. For a fixed generator $g$ of $G$, define the $G$-action on $\Gamma_{16}$ by

$$g = (1, 2, 3)(4, 5, 6) \cdots (3k - 2, 3k - 1, 3k),$$

where $(l, m, n)$ is the cyclic permutation of $(x_l, x_m, x_n)$.

As a basis for $\Gamma_{16}$, we take

$$f_i = \begin{cases} e_i + e_{16}, & (i = 1, \dots, 9), \\ e_i - e_{16}, & (i = 10, \dots, 15), \\ \frac{1}{2}(e_1 + e_2 + \cdots + e_{16}), & (i = 16), \end{cases}$$

where $e_1, \dots, e_{16}$ is the usual orthonormal basis for $\mathbb{R}^{16}$. Then the basis $(f_1, f_2, \dots, f_{16})$ gives required direct splitting. □

The required $G = \mathbb{Z}_3$-action on $(V_{K3}, \Phi_{K3}) \cong 3H \oplus \Gamma_{16}$ is given as follows: Let $G$ act on $3H$ trivially, and on $\Gamma_{16}$, as $\Gamma_{16} \cong \mathbb{Z} \oplus 5\mathbb{Z}[G]$. Then, the conditions REP and GSF in Theorem 2.8 are satisfied. Therefore, we have a locally linear $G$-action with $(m_+, m_-) = (3, 6)$ on a manifold $X'$. Since $X'$ has the even intersection form $(V_{K3}, \Phi_{K3})$, $X'$ is homeomorphic to $K3$ by Freedman's theorem [14]. Thus, we obtain a nonsmoothable locally linear $\mathbb{Z}_3$-action on $K3$ for the smooth structures $E(2)_{k,l}$. Since different pairs of $\{k, l\}$ give different smooth structures on $K3$, the action is nonsmoothable for infinitely many smooth structures.

*Remark* 3.12. By Fintushel-Stern's knot surgery construction [13], we can construct more smooth structures on which our action is nonsmoothable.

*Remark* 3.13. Above, we have constructed a locally linear action of the type $A_1$ in Table 1. By the similar method, locally linear actions of other types can be constructed [25]. Furthermore, among these four types, the types $A_0$ and $B$ can be realized by smooth actions. (See [25] and [33].) On the other hand, we do not know whether the type $A_2$ can be realized by a smooth action, or not.

3(iv). **Remark.** As mentioned in the introduction, many authors have constructed a lot of examples of nonsmoothable locally linear actions [24, 23, 20, 6, 22, 8]. In papers [20, 6, 22], the authors use the gauge theory to prove that the actions are nonsmoothable. It is interesting that these actions are nonsmoothable for *arbitrary* smooth structures.

For instance, [6] and [22] use some $G$-equivariant variants of 10/8-inequalities which give constraints on purely topological data which do not depend on smooth structures. Therefore the locally linear actions which violate these inequalities are clearly nonsmoothable for *arbitrary* smooth structures.

On the other hand, in our case, we need to check the Seiberg-Witten invariant for *each* smooth structure in order to judge the nonsmoothability. This would allude that our result is weaker than results in above papers, however, in another point of view, this fact would suggest that our examples could be *subtle*.

At present, such subtle examples are known only by [8]. In [8], Chen and Kwasik prove that there is a family of symplectic exotic $K3$ surfaces on which every effective action of odd order group is nonsmoothable. On the other hand, we have several examples of smooth actions of odd order cyclic groups on the standard $K3$. This means that there are locally linear actions on $K3$ whose smoothabilities depend on smooth structures.

With these understood, we suggest the following problem (*cf.* Theorem 3.7 and Corollary 3.10).

**Problem 3.14.** Suppose that $n$ is an even positive integer, and $c_{n-2}$ in (1.2) satisfies $c_{n-2} \not\equiv 0 \bmod 3$. Is there a smooth structure on $E(n)$ which admits a smooth $\mathbb{Z}/3$-action with $m_+ > 0$ and $m_- > 0$?

## 4. Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds

In this section, we explain the proof of Theorem 1.10.

Suppose that the standard $\mathbb{Z}_p$-action on $\Sigma(a,b,c)$ extends to a smooth $\mathbb{Z}_p$-action on a smooth contractible 4-manifold $W$ with one fixed point $x$. Then, for some neighborhood $N$ of $x$, $U := (W \setminus N)/\mathbb{Z}_p$ gives a smooth homology cobordism between $Q = \Sigma(a,b,c)/\mathbb{Z}_p$ and a lens space $L(p,q)$. We seek a constraint on such smooth cobordisms. In fact, such a constraint is obtained by using the Seiberg-Witten moduli.

4(i). **The virtual dimension of the Seiberg-Witten moduli.** Let $\Sigma = \Sigma(a_1, \dots a_n)$ be a Seifert integral homology 3-sphere with the Seifert invariant $\{0, (a_1, b_1), \dots, (a_n, b_n)\}$

such that

$$\sum_{i=1}^{n} \frac{b_i}{a_i} = -\frac{1}{\alpha},$$

where $\alpha = a_1 \cdots a_n$. Let $L \to Z$ be the associated $V$-line bundle. Topologically, $Z$ is a 2-sphere.

The disk bundle of a $V$-line bundle $L' \to Z$ is denoted by $D(L')$, and the sphere bundle by $S(L')$. Note that $\Sigma(a_1, \dots, a_n) \cong S(L)$.

Suppose $p$ is an odd prime which is coprime to all $a_i$'s. We concentrate on $(D(L^p), S(L^p))$, and put $V = D(L^p)$ and $Q = S(L^p)$. Note that $Q = S(L^p) \cong \Sigma(a_1, \dots, a_n)/\mathbb{Z}_p$.

Suppose there exists a smooth homology cobordism $U$ between $Q$ and $L(p, q)$. Let us consider the $V$-manifold,

$$X := V \cup U \cup cL(p, q),$$

where $cL(p, q)$ is the cone of $L(p, q)$. For a Spin$^c$-structure $c$ on $X$, let $d(c)$ be the virtual dimension of the Seiberg-Witten moduli for $c$. (The number $d(c)$ is essentially the $w$-invariant for $U$.) The following will give a constraint on smooth cobordisms.

**Proposition 4.1.** *If there exists a smooth homology cobordism $U$ between $Q$ and $L(p, q)$, then $d(c) \leq 0$ for every $c$.*

*Proof.* The proof is based on an argument in the proof of Donaldson's diagonalization theorem by the Seiberg-Witten equations. (See e.g. [27].)

Note that $X$ is negative definite since $c_1(L^p)[Z] < 0$. (See Proposition 4.2.) Since $b_1(X) = 0$, $d(c)$ is odd. Suppose that $d(c) > 0$. By perturbing the Seiberg-Witten equations, the Seiberg-Witten moduli $\mathcal{M}_c$ becomes a $d(c)$-dimensional manifold except the unique reducible, and a neighborhood of the reducible is of the form of a cone of $\mathbb{C}\mathrm{P}^k$ where $k = (d(c) - 1)/2$. Removing the cone from $\mathcal{M}_c$, we obtain a compact manifold $\mathcal{M}'$ whose boundary is $\mathbb{C}\mathrm{P}^k$. On the other hand, $\mathcal{M}'$ is embedded in an infinite dimensional space $\mathcal{B}^*$ of configurations which has the same homotopy type with $\mathbb{C}\mathrm{P}^\infty$. There exists a nonzero cohomology class $u \in H^{2k}(\mathcal{B}^*; \mathbb{Z})$, and it is known that $\langle u, [\mathbb{C}\mathrm{P}^k] \rangle \neq 0$. However $\mathbb{C}\mathrm{P}^k$ bounds a compact manifold $\mathcal{M}'$. This is a contradiction. ☐

4(ii). **Calculation of $d(c)$.** In this subsection, we calculate $d(c)$. Let $\mathrm{Pic}^t_V(Z)$ be the set of isomorphism classes of $V$-line bundle over $Z$. The following facts are easily proved.

**Proposition 4.2** (See e.g. [18]). *The following hold*:

(1) $c_1(L)[Z] = -1/\alpha$ *where* $\alpha = a_1 \cdots a_n$.
(2) $c_1(L^p)[Z] = -p/\alpha$.
(3) $\mathrm{Pic}^t_V(Z)$ *is generated by the class of $L$, and therefore* $\mathrm{Pic}^t_V(Z) \cong \mathbb{Z}$.
(4) $\mathrm{Pic}^t_V(X) \cong \mathrm{Pic}^t_V(D(L^p)) \cong \mathrm{Pic}^t_V(Z) \cong \mathbb{Z}$.
(5) *The restriction map* $\mathrm{Pic}^t_V(X) \to \mathrm{Pic}^t(S(L^p)) \cong \mathrm{Pic}^t(L(p, q))$ *is the mod $p$ map* $\mathbb{Z} \to \mathbb{Z}_p$.

**Proposition 4.3.**

$$c_1(TZ)[Z] = 2 - \sum_{i=1}^{n} \left(1 - \frac{1}{a_i}\right).$$

By above propositions, we have $TZ = L^f$ where

$$f = -\alpha c_1(TZ)[Z] = -\alpha \left(2 - \sum_{i=1}^{n} \left(1 - \frac{1}{a_i}\right)\right).$$

On the other hand, $TL^p = \pi^* L^p \oplus TZ$. Thus the canonical $V$-line bundle of $L^p$ is $K = \pi^* L^{-(f+p)}$.

There is the canonical Spin$^c$-structure on $X$ whose determinant line bundle is $K^{-1}$. We can make another Spin$^c$-structure $c_m$ by twisting the canonical by the $V$-line bundle $\pi^* L^m$ for each $m$. Then the determinant line bundle of $c_m$ is

$$\tilde{L}_m = K^{-1} \otimes \pi^* L^{2m} = \pi^* L^{f+p+2m}.$$

We supposed $p$ is odd. If $f + p + 2m \equiv 0 \bmod p$, then $\tilde{L}_m|_{S(L^p)}$ is trivial, hence $c_m|_{S(L^p)}$ is the spin.

**Theorem 4.4.** *The virtual dimension $d(c_m)$ of the Seiberg-Witten moduli for the* Spin$^c$*-structure $c_m$ is given as follows*:

$$\begin{aligned} d(c_m) = \frac{1}{4}\Bigg[ &-\frac{1}{p\alpha}(f+p+2m)^2 + 1 \\ &- \sum_{i=1}^{n} \frac{1}{a_i} \sum_{l=1}^{a_i - 1} \left\{ \cot\left(\frac{\pi l}{a_i}\right) \cot\left(\frac{\pi p b_i l}{a_i}\right) + 2\cos\left(\frac{\pi(1 + p b_i + 2m b_i) l}{a_i}\right) \csc\left(\frac{\pi l}{a_i}\right) \csc\left(\frac{\pi p b_i l}{a_i}\right) \right\} \\ &- \frac{1}{p} \sum_{l=1}^{p-1} \left\{ \cot\left(\frac{\pi l}{p}\right) \cot\left(\frac{\pi q l}{p}\right) + 2\cos\left(\frac{\pi(1+q+2m')l}{p}\right) \csc\left(\frac{\pi l}{p}\right) \csc\left(\frac{\pi q l}{p}\right) \right\} \Bigg] - 1. \end{aligned}$$

*The integer $m'$ is given as follows: Define an integer $m_s$ by*

$$m_s = \begin{cases} \dfrac{-f}{2} & \text{if } f \text{ is even,} \\ \dfrac{-(f+p)}{2} & \text{if } f \text{ is odd.} \end{cases}$$

*Then $c_{m_s}|_{S(L^p)}$ is the spin. Let $m'_s$ be an integer such that $1 + q + 2m'_s \equiv 0$ mod $p$. (This $m'_s$ corresponds to the spin on $L(p,q)$.) Put $\delta = m - m_s$. Then $m'$ is given by*

$$m' = m'_s + \delta. \tag{4.5}$$

Theorem 4.4 is proved by using Kawasaki's $V$-index theorem [21].

4(iii). **Proof of Theorem 1.10.** Due to Proposition 4.1, there is no smooth homology cobordism between $Q = \Sigma(a, b, c)/\mathbb{Z}_p$ and $L(p, q)$ if one of the following holds:

(1) For some $c_m$, $d(c_m)$ is not an integer.
(2) For some $c_m$, $d(c_m)$ is a positive integer.

For every $(a, b, c)$ and $p$ in Theorem 1.10, it turns out that there is no smooth homology cobordism between $Q = \Sigma(a, b, c)/\mathbb{Z}_p$ and $L(p, q)$ for any $q$ by computer calculations. This means there is no smooth extension of the standard $\mathbb{Z}_p$-action on $\Sigma(a, b, c)$ over $W$.

On the other hand, there is a locally linear extension by Theorem 1.8, and $W$ admits a smooth structure by [7] etc. Thus, the theorem is established.

## References

[1] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. I*, Ann. of Math. **86** (1967), 374–407.
[2] ———, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*, Ann. Math. **88** (1968), 451–491.
[3] M. F. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions*, Essays on topology and related topics, Memoires dédié à George de Rham (ed. A. Haefliger and R. Narashimhan), Springer-Verlag (1970), 18–28.
[4] S. Bauer, *A stable cohomotopy refinement of Seiberg-Witten invariants. II*, Invent. Math. **155** (2004), no. 1, 21–40.
[5] S. Bauer and M. Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants. I*, Invent. Math. **155** (2004), 1–19.
[6] J. Bryan, *Seiberg-Witten theory and* $\mathbb{Z}/2^p$ *actions on spin* 4*-manifolds*, Math. Res. Lett. **5** (1998), no. 1-2, 165–183.
[7] A. J. Casson and J. L. Harer, *Some homology lens spaces which bound rational homology balls*, Pacific J. Math. **96** (1981), no. 1, 23–36.
[8] W. Chen and S. Kwasik, *Symmetries and exotic smooth structures on a* $K3$ *surface*, preprint.
[9] A. L. Edmonds, *Construction of group actions on four-manifolds,* Trans. Amer. Math. Soc. **299** (1987), no. 1, 155–170.
[10] A. L. Edmonds and J. H. Ewing, *Realizing forms and fixed point data in dimension four*, Amer. J. Math. **114** (1992), 1103–1126.
[11] F. Fang, *Smooth group actions on* 4*-manifolds and Seiberg-Witten invariants*, International J. Math. **9**, No.8 (1998) 957–973.
[12] R. Fintushel and R. Stern, *Pseudofree orbifolds*, Ann. of Math. **122** (1985), no. 2, 335–364.
[13] ———, *Knots, links and* 4*-manifolds*, Invent. math. **134** (1998), 363–400.
[14] M. H. Freedman and F. Quinn, Topology of 4-manifolds, Princeton Mathematical Series, 39. Princeton University Press, Princeton, NJ, 1990.
[15] Y. Fukumoto and M. Furuta, *Homology 3-spheres bounding acyclic 4-manifolds*, Math. Res. Lett. **7** (2000), no. 5-6, 757–766.
[16] Y. Fukumoto and N. Nakamura, in preparation.
[17] M. Furuta, Y. Kametani and N. Minami, *Stable-homotopy Seiberg-Witten Invariants for Rational Cohomology* $K3\#K3$*'s*, Journal of Mathematical Sciences, The University of Tokyo **8** (2001), No. 1, 157-176.
[18] M. Furuta and B. Steer, *Seifert fibred homology* 3*-spheres and the Yang-Mills equations on Riemann surfaces with marked points*, Adv. Math. **96** (1992), no. 1, 38–102.
[19] R. E. Gompf and A. I. Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999.

[20] I. Hambleton and R. Lee, *Smooth group actions on definite 4-manifolds and moduli spaces*, Duke. Math. J. **78**, No.3 (1995), 715–732.
[21] T. Kawasaki, *The index of elliptic operators over V-manifolds*, Nagoya Math. J. **84** (1981), 135–157.
[22] K. Kiyono, *Examples of unsmoothable group actions on* $\overset{n}{\sharp} S^2 \times S^2$, in preparation.
[23] S. Kwasik and T. Lawson, *Nonsmoothable* $\mathbb{Z}_p$ *actions on contractible 4-manifolds*, J. Reine. Angew. Math. **437** (1993), 29–54.
[24] S. Kwasik and K. B. Lee, *Locally linear actions on 3-manifolds*, Math. Proc. Camb. Phil. Soc. **104**, (1988), 253–260.
[25] X. Liu and N. Nakamura, *Pseudofree* $\mathbb{Z}/3$*-actions on* $K3$ *surfaces*, Proc. Amer. Math. Soc. **135** (2007), no. 3, 903–910.
[26] ———, *Unsmoothable group actions on elliptic surfaces*, preprint.
[27] J. D. Moore, Lectures on Seiberg-Witten invariants, Lecture Notes in Mathematics, 1629. Springer-Verlag, Berlin, 1996.
[28] N. Nakamura, *Mod p vanishing theorem of Seiberg-Witten invariants for 4-manifolds with* $\mathbb{Z}_p$*-actions*, Asian J. Math. **9**, (2006), no.4, 731–748.
[29] ———, *Bauer-Furuta invariants under* $\mathbb{Z}_2$*-actions*, preprint, math.GT/0705.1595.
[30] ———, *Bauer-Furuta invariants and a non-smoothable involution on* $K3\#K3$, The 54th Topology Symposium.
[31] P. Shanahan, The Atiyah-Singer index theorem. An introduction. Lecture Notes in Mathematics, 638. Springer, Berlin, 1978.
[32] M. Ue, *On the topology of elliptic surfaces–a survey*, Amer. Math. Soc. Transl. **160** (1994), 95–123.
[33] D.-Q. Zhang, *Logarithmic Enriques surface*. J. Math. Kyoto Univ. **31** (1991), 419–466.

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