Edmonds-Ewing の理論

Freedman 理論の群作用への応用

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Introduction

Symmetries of manifolds $\longleftrightarrow$ Group actions

- X: smooth manifold $\Rightarrow$ smooth action on X
- X: topological manifold $\Rightarrow$ topological action on X

Locally linear action

- G: finite group
- X^n: n-dim. TOP manifold

Definition

A topological G-action on X is locally linear if
$\forall x \in X$, $\exists U_x$: G_x-invariant nbd. of x s.t.
(G_x = the isotropy subgroup of x)

- $U_x \underset{homeo}{\cong} \mathbb{R}^n$,
- $G_x \curvearrowright \mathbb{R}^n \cong U_x$ is orthogonal linear.

In general, smooth $\underset{\not\Leftarrow}{\Rightarrow}$ locally linear

Locally linear actions in dimension 4

Existence

- [Edmonds-Ewing'92]
  $\pi_1 X = 1$, $G = \mathbb{Z}_p$ (p: prime), X^G: isolated のとき, 固定点データ & G 同変な交叉形式が実際の X 上の 局所線型 G 作用として実現できるための必要十分条件を与えた.

Classification

- [Wilczyński'94],[Wilczyński-Bauer'96]
  $\pi_1 X = 1$, $G = \mathbb{Z}_m$ ($m \in \mathbb{Z}_{\geq 2}$), X^G: isolated として, X 上の局所線型 G 作用があったとき, それと同じ固定点データ & 交叉形式を持つ作用がどのくらいあるかを分類した.

Cf. Nonsmoothable actions

- [Liu-N.'07-08]
  ∃Nonsmoothable loc. lin. $\mathbb{Z}_p$-actions on elliptic surfaces
  $(p = 3, 5, 7)$
- [N.'08] ∃Nonsmoothable loc. lin. involution on $K3\#K3$
- [Kiyono '08] $\forall X$: spin, $\neq S^4$, $S^2 \times S^2$, $\exists p$: large prime,
  ∃Nonsmoothable loc. lin. $\mathbb{Z}_p$-actions on X.

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Locally linear actions in dimension 4

以下, 次の論文の紹介をする.

- [Edmonds-Ewing'92]
  A.L.Edmonds and J.H.Ewing,
  Realizing forms and fixed point data in dimension four,
  Amer. J. Math. **114** (1992), 1103–1126.
- [Edmonds'87]
  A.L.Edmonds,
  Construction of group actions of four-manifolds,
  Trans. AMS. **299** (1987), 155–177.

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The result of Edmonds-Ewing

Let X be a TOP closed 4-manifold with $\pi_1 X = 1$, and $G = \mathbb{Z}_p$ (p: odd prime).
Suppose $\exists$loc. lin. $G \curvearrowright X$ s.t. X^G: discrete.

- For each $x \in X^G$, $U_x \cong \mathbb{C}_a \oplus \mathbb{C}_b$, where $\mathbb{C}_k$ is the complex 1-dim. representation of weight k.
  $\longrightarrow$ we say the type of x is (a, b).
  If $X^G = \{x_0, x_1, \ldots, x_{n+1}\}$,
  $\Rightarrow$ fixed point data $D = \{(a_0, b_0), \ldots, (a_{n+1}, b_{n+1})\}$.
- $G \curvearrowright H^2(X; \mathbb{Z})$, i.e., $V = H^2(X; \mathbb{Z})$ is a $\mathbb{Z}[G]$-module, and the intersection form $\Psi \colon V \times V \to \mathbb{Z}$ is G-invariant.
  $\forall g \in G$, $\Psi(gx, gy) = \Psi(x, y)$.

Conversely, suppose $D = \{(a_0, b_0), \dots, (a_{n+1}, b_{n+1})\}$ and G-invariant $\Psi\colon V \times V \to \mathbb{Z}$ are given.

Question

When D and (V, Ψ) are realized by a loc. lin. action on a 4-manifold?

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Statement of the result

Theorem A [EE '92]

$G = \mathbb{Z}_p$ (p: odd prime).
For $D = \{(a_0, b_0), \dots, (a_{n+1}, b_{n+1})\}$ and $\Psi\colon V \times V \to \mathbb{Z}$,

- REP: $V = n\mathbb{Z} \oplus F$ as $\mathbb{Z}[G]$-module, F: free $\mathbb{Z}[G]$-module.
- GSF:
  $$\mathrm{Sign}(g, (V, \Psi)) = \sum_{i=0}^{n+1} \frac{\zeta^{a_i}+1}{\zeta^{a_i}-1}\frac{\zeta^{b_i}+1}{\zeta^{b_i}-1}, \qquad \zeta = \exp\frac{2\pi\sqrt{-1}}{p}$$
  $\Psi \otimes \mathbb{R} \Rightarrow V \otimes \mathbb{R} = V_+ \oplus V_-$ (V_+ V_-: G-modules),
  $\mathrm{Sign}(g, (V, \Psi)) = \mathrm{tr}(g|V_+) - \mathrm{tr}(g|V_-)$.
- TOR: (Certain condition related to Reidemeister torsion.)

REP, GSF and TOR $\Leftrightarrow$ $\exists X^4$ with $\pi_1 X = 1$ and $\exists$ loc. lin. action $G \curvearrowright X$ which realizes D and (V, Ψ).

Remark: $p = 2$ のときの定理もある.

Outline of the proof

The proof of “$\exists \mathbb{Z}_p$-action $\Rightarrow$ REP, GSF,TOR”

Suppose the action is given.

- REP $\Leftarrow$ Lefschetz formula + [Edmonds’89].
- GSF $\Leftarrow$ [Wall] For loc. lin. $\mathbb{Z}_p$-action, the G-signature formula holds.
- TOR については省略.

The proof of “REP, GSF,TOR $\Rightarrow \exists \mathbb{Z}_p$-action”

Construction of the action $\rightarrow$ equivariant handle construction

$$V = n\mathbb{Z} \oplus F, \quad \Psi \colon V \times V \to \mathbb{Z}$$

Step 0

$$\begin{aligned} &\text{0-handle} && B_0^4 \subset \mathbb{C}_{a_0} \oplus \mathbb{C}_{b_0}, \\ n\mathbb{Z} \leftrightarrow\ &G\text{-inv. 2-handles} && H_i = D^2 \times D^2 \subset \mathbb{C}_{a_i} \oplus \mathbb{C}_{b_i}, \\ F \leftrightarrow\ &\text{free 2-handles} && \end{aligned}$$

Step 1 Represent (V, Ψ) by a $\mathbb{Z}_p$-invariant framed link L on ∂B_0.

Step 2 Attach $H_1, \ldots, H_n$ & free handles to B_0 along L.
$\rightarrow \mathbb{Z}_p \curvearrowright B_0 \cup H_1 \cup \cdots H_n \cup$ (free handles).

Step 3 Note that

- $\Sigma := \partial(B_0 \cup H_1 \cup \cdots H_n \cup (\text{free handles}))$ is a $\mathbb{Z}$-homology 3-sphere.
- $\mathbb{Z}_p \curvearrowright \Sigma$ freely

Theorem B [EE]

GSF & TOR $\Rightarrow$ $\exists$loc. lin. $\mathbb{Z}_p$-action on $\exists W^4$ s.t.

- W: contractible, $\partial W = \Sigma$.
- $\mathbb{Z}_p \curvearrowright W$ extends $\mathbb{Z}_p \curvearrowright \Sigma = \partial W$.
- $W^{\mathbb{Z}_p} = \{1\text{pt}\}$ $\rightarrow$ the type (a_{n+1}, b_{n+1})

Then,

$$\mathbb{Z}_p \curvearrowright X := (B_0 \cup H_1 \cup \cdots H_n \cup (\text{free handles})) \cup_\Sigma W.$$

Proof of Theorem B

The proof of Theorem B uses Surgery theory + Freedman theory.

The idea of the proof

- $Q := \Sigma/\mathbb{Z}_p$ $\leftarrow$ "homology lens space"
- Construct a cobordism V between Q and some $L = L(p,q)$ s.t. V homotopy equivalent to $L \times I$.
- Let $\tilde{V}$ be the univ. cover of V. Then $\partial \tilde{V} = \Sigma \cup S^3$. Cap $\tilde{V}$ by a 4-ball B with a linear $\mathbb{Z}_p$-action. $\rightarrow$ $W = \tilde{V} \cup B^4$

Construction of the cobordism

- To construct the cobordism V, we will need a deg$=1$ map $f\colon Q=\Sigma/\mathbb{Z}_p\to L$.
- Q is a homology lens.
  $\to H^k := H^k(Q;\mathbb{Z}_p)=\mathbb{Z}_p$ $(0\le k\le 3)$.

Lemma 1

Fix a generator $x\in H^1(Q;\mathbb{Z}_p)=\mathbb{Z}_p$.
$\Rightarrow \exists q(\neq 0)\in\mathbb{Z}_p, \exists f\colon Q\to L=L(p,q)$, deg$=1$.

Lemma 1

Fix a generator $x\in H^1(Q;\mathbb{Z}_p)=\mathbb{Z}_p$.
$\Rightarrow \exists q(\neq 0)\in\mathbb{Z}_p, \exists f\colon Q\to L=L(p,q)$, deg$=1$.

Proof

- In this case, Bockstein $\beta\colon H^1(Q;\mathbb{Z}_p)\to H^2(Q;\mathbb{Z}_p)$ is an isomorphism.
  $\Rightarrow \beta x,\ x\cup\beta x$ are generators of $H^2,\ H^3$.
- Let $[Q]:=P.D.[1pt]\in H^3$.
  $\Rightarrow \exists q(\neq 0)\in\mathbb{Z}_p,\ x\cup\beta x=q[Q]$.
- Note $H^1(Q;\mathbb{Z}_p)\cong[Q,K(\mathbb{Z}_p,1)]\cong\mathbb{Z}_p$ &
  $\exists$surjection $[Q,L(p,q)]\cong\mathbb{Z}\to[Q,K(\mathbb{Z}_p,1)]\cong\mathbb{Z}_p$.
- In fact, $x\in H^1\cong[Q,K(\mathbb{Z}_p,1)]$ has a lift $\tilde{x}\in[Q,L(p,q)]$ represented by $f\colon Q\to L=L(p,q)$ s.t.
  - $\exists y\in H^1(L;\mathbb{Z}_p),\ y\cup\beta y=q[L]$.
  - $x=f^*y$.

- Then $q[Q] = x \cup \beta x = f^*(y \cup \beta y) = f^*(q[L]) = qf^*[L]$.
  $\therefore [Q] \equiv f^*[L] \bmod p$. $\therefore \deg f \equiv 1 \bmod p$.
- To change the degree by a multiple of p, use

$$Q = Q \# S^3 \to Q \vee S^3 \to L \vee L \to L.$$

(証明おわり)

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For Q, L as above, we will define invariants:

- α-invariants $\alpha(g, Q)$, $\alpha(g, L)$,
- Reidemeister torsion $\Delta(Q)$, $\Delta(L) \in \mathbb{Q}[\zeta]^\times$, $\zeta = \exp \frac{2\pi\sqrt{-1}}{p}$.

Then, Theorem B will be proved from the following:

Proposition

$\exists \mathbb{Z}[\mathbb{Z}_p]$-homology cobordism $(V; Q, L)$ between Q and $L = L(p, q)$ s.t. $\pi_1 V = \mathbb{Z}_p$ if and only if $\alpha(g, Q) = \alpha(g, L)$, $\Delta(Q) \sim u^2 \Delta(L)$, where

- $\exists u$: an image in $\mathbb{Q}[\zeta]$ of a unit of $\mathbb{Z}[\mathbb{Z}_p]$. $(\mathbb{Z}[\mathbb{Z}_p] \to \mathbb{Z}[\zeta] \to \mathbb{Q}[\zeta])$
- $\Delta_1 \sim \Delta_2 \Leftrightarrow \Delta_1 = \pm \zeta^i \Delta_2$.

Remark

$(V; Q, L)$ is a $\mathbb{Z}[\mathbb{Z}_p]$-homology cobordism
$\overset{def}{\Leftrightarrow} H_*(V, Q; \mathbb{Z}[\mathbb{Z}_p]) = H_*(V, L; \mathbb{Z}[\mathbb{Z}_p]) = 0$.

α-invariant

- $\Sigma \overset{\mathbb{Z}_p}{\to} Q$ is classified by $Q \to K(\mathbb{Z}_p, 1)$.
- [Fact] $\Omega_3(K(\mathbb{Z}_p, 1)) \otimes \mathbb{Q} = 0$.
  $\Rightarrow \exists r$, rQ bounds over $K(\mathbb{Z}_p, 1)$.
  $\Rightarrow \exists(\mathbb{Z}_p \curvearrowright X^4)$ free s.t. $\partial X = r\Sigma$.
- The intersection form $\Psi\colon H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$ is $\mathbb{Z}_p$-invariant.

Definition

For $g \neq 1$,

$$\alpha(g, Q) = \frac{\mathrm{Sign}(g, X)}{r}.$$

$\Psi \otimes \mathbb{R} \Rightarrow H_2(X) \otimes \mathbb{R} = H_+ \oplus H_-$, $\mathrm{Sign}(g, X) = \mathrm{tr}(g|H_+) - \mathrm{tr}(g|H_-)$.

- Suppose $\exists$loc. lin. $\mathbb{Z}_p \curvearrowright X^4$ with $\partial X = \Sigma$ extending $\mathbb{Z}_p \curvearrowright \Sigma = \partial X$. Not necessarily free.

$\Rightarrow$ $\alpha(Q)$ can be calculated from G-signature formula.

$$\alpha(Q) = \mathrm{Sign}(g, X) + 4\sum_k e(F_k)\frac{\zeta^{c_k}}{(\zeta^{c_k} - 1)^2} + \sum_i \frac{\zeta^{a_i} + 1}{\zeta^{a_i} - 1}\frac{\zeta^{b_i} + 1}{\zeta^{b_i} - 1}.$$

Ex.

- $\mathbb{Z}_p \curvearrowright S^3 \subset \mathbb{C}_a \oplus \mathbb{C}_b$, $L(p; (a, b)) = S^3/\mathbb{Z}_p$.

$$\Rightarrow \alpha(L(p; (a, b)) = \frac{\zeta^a + 1}{\zeta^a - 1}\frac{\zeta^b + 1}{\zeta^b - 1}.$$

Reidemeiseter torsion

- Fix a cell structure on Q $\to$ induced cell structure on Σ.
  $\Rightarrow$ $C_*(\Sigma)$ is a free $\mathbb{Z}[\mathbb{Z}_p]$-module.

Lemma 2 (Milnor)

$C_*(\Sigma) \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\zeta]$ is acyclic.

Proof

- Note $\mathbb{Q}[\mathbb{Z}_p] \cong \mathbb{Q} \oplus \mathbb{Q}[\zeta]$.
- $H_*(C_* \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\mathbb{Z}_p]) \cong H_*(C_* \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}) \oplus H_*(C_* \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\zeta])$
- By definition, $H_*(C_*(\Sigma) \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\mathbb{Z}_p]) \cong H_*(\Sigma; \mathbb{Q})$
- Note $\mathbb{Z}_p \curvearrowright H_*(\Sigma; \mathbb{Q})$ trivially.
- If $H_*(C_* \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\zeta]) \neq 0$, $\mathbb{Z}_p$ acts on it nontrivially.
  $\Rightarrow$ $H_*(C_* \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\zeta])$ must be 0.

- Then, Torsion invariant of $C_*(\Sigma) \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\zeta]$ is defined as $\Delta(Q) \in \mathbb{Q}[\zeta]^\times$.
- $\Delta(Q)$ depends on cell str. & basis.
- Define $\Delta_1 \sim \Delta_2 \overset{def}{\Leftrightarrow} \Delta_1 = \pm\zeta^i \Delta_2$.
  Then the equivalence class of $\Delta(Q)$ does not depend on cell str. & basis.

Recall:

Proposition

$\exists\mathbb{Z}[\mathbb{Z}_p]$-homology cobordism $(V; Q, L)$ between Q and $L = L(p,q)$ s.t. $\pi_1 V = \mathbb{Z}_p$ if and only if $\alpha(g,Q) = \alpha(g,L)$, $\Delta(Q) \sim u^2\Delta(L)$.

The proof of "$\exists(V; Q, L) \Rightarrow \alpha(g,Q) = \alpha(g,L)$, $\Delta(Q) \sim u^2\Delta(L)$"

- Suppose $\exists\mathbb{Z}[\mathbb{Z}_p]$-cobordism $(V; Q, L)$ s.t. $\pi_1 V = \mathbb{Z}_p$.
- Let $\tilde{V}$ be the univ. cover of V.

$$\begin{aligned}\alpha(g,Q) - \alpha(g,L) &= \mathrm{Sign}(g,\tilde{V}) \ (\because \mathbb{Z}_p \curvearrowright \tilde{V} \text{ freely}) \\ &= 0 \ (\because H_2(\tilde{V}) = 0)\end{aligned}$$

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- Let $\Delta(V;Q)$ be the torsion of $C_*(\tilde{V},\Sigma) \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\zeta]$.
- Let $\Delta(V;L)$ be the torsion of $C_*(\tilde{V},S^3) \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \mathbb{Q}[\zeta]$.

$\Rightarrow$ [Milnor's duality] $\Delta(V,Q) \sim (\overline{\Delta(V,L)})^{-1}$

- $0 \to C_*(\Sigma) \to C_*(\tilde{V}) \to C_*(\tilde{V},\Sigma) \to 0$
  $\Rightarrow \Delta(V) \sim \Delta(V,Q)\Delta(Q)$.
- $0 \to C_*(S^3) \to C_*(\tilde{V}) \to C_*(\tilde{V},S^3) \to 0$
  $\Rightarrow \Delta(V) \sim \Delta(V,L)\Delta(L)$.
- Let $u = \Delta(V,L) \sim (\overline{\Delta(V,Q)})^{-1}$. $\Rightarrow \Delta(Q) \sim \bar{u}u\Delta(L)$.

Note u comes from $\mathbb{Z}[\mathbb{Z}_p]$. $\Rightarrow u = \bar{u}$.

(前半の証明終)

Surgery exact sequence of 4-dim. TOP

- X: 4-dim. Poincaré complex, $\pi = \pi_1 X$.

$$L_5(\mathbb{Z}[\pi]) \to S(X) \to N(X) \xrightarrow{\sigma} L_4(\mathbb{Z}[\pi]).$$

- $L_*(\mathbb{Z}[\pi]) = L_*^h(\mathbb{Z}[\pi])$.
- $S(X) = S^h_{TOP}(X)$
  $= \{(M, f) \mid M^4 : \text{TOP mfd}, f\colon M \to X \text{ h.e. }\}/\text{h-cobordism}$
- $N(X) = N_{TOP}(X)$
  $= \{(f, b)\colon (M, \nu_M) \to (X, \eta) \text{ deg=1, normal }\}/\text{normal cob}$
  $\cong [X, G/TOP]$
  - ν_M: stable normal TOP (micro)bundle of M.
  - X: Poincaré cpx $\Rightarrow \exists! \nu_X$: Spivak normal fibration.
  - η: a TOP bundle reduction of ν_x.

The proof of "$\alpha(g, Q) = \alpha(g, L)$, $\Delta(Q) \sim u^2\Delta(L) \Rightarrow \exists(V; Q, L)$"

- By Lemma 1, $\exists f\colon Q \to L$, deg $= 1$.
- Q, L: parallelizable $\Rightarrow$ f は normal map と思える.
- [Fact] p: odd $\Rightarrow [L, G/TOP] = 0$.

$$\Rightarrow N(L) = \{1pt\}.$$

- $\mathrm{id}_L\colon L \to L$ is another deg$= 1$ map.

$\Rightarrow$ $\exists$ normal cobordism $F\colon (V; Q, L) \to L \times (I; \{0\}, \{1\})$ s.t. $F|_Q = f$, $F|_L = \mathrm{id}_L$.

- V 内の circle たちの上で surgery $\Rightarrow \pi_1 V = \mathbb{Z}_p$ にできる.
- さらに surgery で F をホモトピー同値にするための obstruction:

$$\sigma(F) \in L_4^h(\mathbb{Z}[\mathbb{Z}_p]).$$

Fact [Bak,Wall]

- p : odd $\Rightarrow L_4^h(\mathbb{Z}[\mathbb{Z}_p]) \cong \bigoplus_{\frac{p+1}{2}} \mathbb{Z} \oplus$ (2-primary torsion)
- The free part of $\sigma(F)$ is multisignature.

Fact [Edmonds]

- (The 2-torsion part of $\sigma(F)) = 0 \Leftrightarrow \Delta(Q) \sim u^2\Delta(L)$.

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Multisignature

- Let $\tilde{V}$ be the univ. cover of $V \Rightarrow \partial\tilde{V} = \Sigma \sqcup S^3$.
- Then, the intersection form $\Psi\colon H_2(\tilde{V};\mathbb{Z}) \times H_2(\tilde{V};\mathbb{Z}) \to \mathbb{Z}$ is a $\mathbb{Z}_p$-invariant form.
- $\Psi \otimes \mathbb{R} \Rightarrow H_2(\tilde{V}) \otimes \mathbb{R} = H_+ \oplus H_-$.
  Note H_+, H_- are $\mathbb{Z}_p$-modules.
- The multisignature of Ψ is given by

$$\mathrm{Sign}(\mathbb{Z}_p, \tilde{V}) = [H_+] - [H_-] \in RO(\mathbb{Z}_p) \cong \bigoplus_{\frac{p+1}{2}} \mathbb{Z}.$$

- $\mathrm{Sign}(\mathbb{Z}_p, \tilde{V})$ can be calculated from $\mathrm{Sign}(g, \tilde{V})$, $g \in \mathbb{Z}_p$.

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- Suppose $\alpha(g, Q) = \alpha(g, L)$ for $g \neq 1$.

$\Rightarrow$ $\mathrm{Sign}(g, \tilde{V}) = \alpha(g, Q) - \alpha(g, L) = 0$ for $g \neq 1$.

$\Rightarrow$ $\mathrm{Sign}(\mathbb{Z}_p, \tilde{V})$ は正則表現のいくつかの和 &
$\mathrm{Sign}(\tilde{V}) = \mathrm{Sign}(1, \tilde{V})$ は p の倍数.

$\Rightarrow$ $\tilde{V}$ に $|\mathrm{Sign}(\tilde{V})|$ 個の $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ を $\mathbb{Z}_p$ 同変に連結和してやると $\mathrm{Sign} = 0$ にできる.

$\Rightarrow$ (The free part of the surgery obstruction $\sigma(F)$)$= 0$.

- 一方, $\Delta(Q) \sim u^2\Delta(L) \Rightarrow$ (The torsion part of $\sigma(F)$)$= 0$.

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まとめると

$$\boxed{\begin{matrix}\alpha(g, Q) = \alpha(g, L) \\ \Delta(Q) \sim u^2\Delta(L)\end{matrix}} \Rightarrow \sigma(F) = 0 \Rightarrow \boxed{V \text{ を surgery で } L \times I \text{ に ホモトピー同値にできる.}}$$

- この V が求めるもの.

(証明おわり)

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In fact,

$$\text{GSF, TOR} \Rightarrow \boxed{\begin{array}{c}\alpha(g,Q)=\alpha(g,L)\\ \Delta(Q)\sim u^2\Delta(L)\end{array}}$$

Theorem B follows from the following:

Theorem C

Suppose a free $\mathbb{Z}_p$-action on Σ: $\mathbb{Z}HS^3$ is given.
$\exists$loc. lin. action $\mathbb{Z}_p \curvearrowright \exists W$ which extends $\mathbb{Z}_p \curvearrowright \Sigma$ s.t.

- W: contractible, $\partial W = \Sigma$,
- $W^{\mathbb{Z}_p} = \{1pt\}$

if and only if $\alpha(g,Q)=\alpha(g,L)$, $\Delta(Q)\sim u^2\Delta(L)$.

The proof of Theorem C, part 1

$\boxed{\Rightarrow}$

- Suppose $\exists$loc. lin. extention $(\mathbb{Z}_p \curvearrowright W)$ s.t. $W^{\mathbb{Z}_p} = \{x_0\}$.
- Let N be a $\mathbb{Z}_p$-inv. nbd of x_0, and $U := W \setminus N$.

$\Rightarrow$ $\mathbb{Z}_p \curvearrowright U$ freely & $U/\mathbb{Z}_p$ is a $\mathbb{Z}[\mathbb{Z}_p]$-homology cobordism between Q & L.

$\Rightarrow$ By Proposition, $\alpha(g,Q)=\alpha(g,L)$, $\Delta(Q)\sim u^2\Delta(L)$.

The proof of Theorem C, part 2

⇐

- Suppose $\alpha(g, Q) = \alpha(g, L)$, $\Delta(Q) \sim u^2\Delta(L)$.
- By Proposition, $\exists\mathbb{Z}[\mathbb{Z}_p]$-h.cob. $(V; Q, L)$ s.t. $V \underset{h.e.}{\simeq} L \times I$.
- Let $\tilde{V}$ be the univ. cover of $V \Rightarrow \mathbb{Z}_p \curvearrowright (\tilde{V}; \Sigma, S^3)$.
- 片方の境界 $\mathbb{Z}_p \curvearrowright S^3$ を ∃linear action $\mathbb{Z}_p \curvearrowright B^4$ でフタをする.

$\Rightarrow W = \tilde{V} \cup B^4$

- $V \underset{h.e.}{\simeq} L \times I$ なので $\tilde{V} \underset{h.e.}{\simeq} S^3 \times I$. $\Rightarrow W$: contractible.

(証明おわり)

おわりに...

By Theorem B or C, under some conditions, free $\mathbb{Z}_p$-actions on homology spheres Σ can be extended loc. linearly to contracrible W s.t. $\partial W = \Sigma$.

Question

Are such loc. lin. actions smoothable?

[Kwasik-Lawson'92]

∃Nonsmoothable actions when Σ are Brieskorn.

- [Fukumoto-N.'07-] Alternative proof by Seiberg-Witten.
- *Cf.* [Akbulut-Yasui] Cork.

Problem

Find smoothable examples.