Smoothability of $\mathbb{Z} \times \mathbb{Z}$-actions on 4-manifolds

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Main results

- E: Enriques surface, $\rightarrow$ non-spin, $\pi_1 \cong \mathbb{Z}/2$, $E_8 \oplus H$, $b^+ = 1$
- $X := E\#(S^2 \times S^2)$.

We will explain:
$\exists$Nonsmoothable $\mathbb{Z} \times \mathbb{Z}$-action on X s.t. each of the generators is smoothable.

Main Theorem

There exist two self-homeomorphisms $f_1, f_2 \colon X \xrightarrow{\cong} X$ s.t.

1. f_1 and f_2 commute: $f_1 \circ f_2 = f_2 \circ f_1$.
2. Each one of f_1 and f_2 can be smoothed for some smooth structure on X.
3. However, f_1 and f_2 can not be smoothed at the same time for any smooth structure on X.

We will also talk about

Theorem 2

Let Y be an Enriques surf.
Then, $\exists$ self-homeomorphism $f\colon Y \to Y$ which is nonsmoothable for any smooth structure on Y.

The strategy of proofs

The proofs will be divided into 2-steps:

- Give constraints on diffeomorphisms.
  $\to$ Seiberg-Witten gauge theory on families.
- Construct homeomorphisms which violate the constraints.
  $\to$ Use a result of Hambleton-Kreck that an Enriques surface E has a topological splitting:

$$E \underset{\text{homeo.}}{\cong} E' \# (S^2 \times S^2).$$

The statement of results
- Main results
- The strategy of proofs

Proof of Theorem 2
- Constraint on diffeomorphisms
- Construction of a nonsmoothable homeomorphism

Proof of Main theorem
- Constraint on pairs of diffeomorphisms
- Construction of a nonsmoothable action

Seiberg-Witten moduli spaces for families
- Proof of Proposition 1
- Proof of Proposition 2

Proof of Theorem 2

Part 1: Constraint on diffeomorphisms

By Seiberg-Witten gauge theory, we can prove:

Proposition 1

- Y: a smooth 4-manifold homeo. to an Enriques surface.
- c: a Spin^c-structure on Y whose c_1 is a torsion class.
- $f\colon Y \to Y$, an orintation preserving diffeomorphism.

If $f^*c \cong c$, then f preserves the orientation of $H^+(Y;\mathbb{R})$.

Cf. [Donaldson]
Every ori. pres. diffeo. of $K3$ preserves the ori. of $H^+(K3;\mathbb{R})$.

Proof of Theorem 2

Part 2: Construction of a nonsmoothable homeomorphism

By Proposition 1, a homeomorphism of an Enriques surf. $f\colon Y \to Y$ is nonsmoothable if

- $f^*c \cong c$, where c is a torsion Spin^c-structure,
- f reverses the ori. of $H^+(Y)$.

Theorem [Hambleton-Kreck'88]

An Enriques surface is homeomorphic to $\Sigma \# |E_8| \# (S^2 \times S^2)$, where

- Σ is a nonspin rational homology 4-sphere with $\pi_1 = \mathbb{Z}/2$.
- $|E_8|$ is a simply-connected topological 4-manifold whose intersection form is the negative definite E_8.

Remark

Neither Σ nor $|E_8| \# (S^2 \times S^2)$ is smoothable.

Construction of a nonsmoothable homeomorphism

Step 1. Choose an ori. pres. self-diffeo. $\varphi\colon S^2 \times S^2 \to S^2 \times S^2$ s.t.

- $\exists$4-ball $B_0 \subset S^2 \times S^2$ s.t. $\varphi|_{B_0} = \mathrm{id}$.
- φ reverses the ori. of $H^+(S^2 \times S^2)$.

Ex. Assume $S^2 \times S^2 = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. Let φ_0 be the complex conjugation on $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. To obtain a required φ, perturb φ_0 around a fixed point.

Step 2. Define a self-homeomorphism f of $\Sigma\#|E_8|\#(S^2 \times S^2)$ by

$$f = (\mathrm{id}_{\Sigma\#|E_8|})\#\varphi.$$

Note that f reverses the ori. of H^+.
Then, Theorem 2 is proved by the following:

Claim

For a torsion Spin^c-structure c, $f^*c \cong c$.
Here, c is assumed as a topological Spin^c-structure.

Proof

- Note $c = c'\#c_0$, where
  - c' is a torsion Spin^c str. on $\Sigma\#|E_8|$, and
  - c_0 is the unique spin str. on $S^2 \times S^2$.
- $f|_{\Sigma\#|E_8|} = \mathrm{id}_{\Sigma\#|E_8|}$ fixes c'.
- $f|_{S^2\times S^2}$ preserves c_0. □

Proof of Main theorem

Part 1: Constraint on pairs of diffeomorphisms

- Let X be a smooth 4-manifold homeo. to $E\#(S^2 \times S^2)$, where E: Enriques.
- Suppose two diffeo. $f_1, f_2 \colon X \to X$ s.t $f_1 \circ f_2 = f_2 \circ f_1$ are given.

$\Rightarrow$ Can construct a "double mapping torus" $X_{(f_1,f_2)} \to T^2$ as

$$X_{(f_1,f_2)} = X \times [0,1] \times [0,1]/(f_1, f_2).$$

- Choose a smooth family of metrics $\{g_b\}_{b \in T^2}$ on $X_{(f_1,f_2)}$, where g_b is a Riemannian metric on the fiber X_b over $b \in T^2$.

The statement of results
Proof of Theorem 2
Proof of Main theorem
Seiberg-Witten moduli spaces for families
Constraint on pairs of diffeomorphisms
Construction of a nonsmoothable action

- Define an $\mathbb{R}^2$-vector bundle $H^+_{(f_1,f_2)} \to T^2$ by

$$H^+_{(f_1,f_2)} = \coprod_{b \in T^2} H^{+g_b},$$

where H^{+g_b} is the space of g_b-self-dual harmonic 2-forms on X_b.

Roughly speaking,

$$H^+_{(f_1,f_2)} = H^+(X) \times [0,1] \times [0,1]/(f_1^*, f_2^*).$$

Nobuhiro Nakamura
Smoothability of $\mathbb{Z} \times \mathbb{Z}$-actions on 4-manifolds

Proposition 2

Let c be a torsion Spin^c-structure on X.
If $f_i^* c \cong c$ for $i = 1, 2$, then

$$w_2\left(H^+_{(f_1,f_2)}\right) = 0.$$

Remark Proposition 1 can be stated as:

Proposition 1′ If $f^* c \cong c$, then $w_1(H^+_f) = 0$.

(Roughly, $H^+_f = H^+(Y, \mathbb{R}) \times [0,1]/f^*$.)

Part 2: Construction of a nonsmoothable $\mathbb{Z} \times \mathbb{Z}$-action

- For $i = 1, 2$, let (S_i, φ_i) be copies of $(S^2 \times S^2, \varphi)$.
- Let $X := S_1 \# (\Sigma \# |E_8|) \# S_2$.
- Note that $(\Sigma \# |E_8|) \# S_i$ $(i = 1, 2)$ is homeomorphic to an Enriques surf E.
- Then, X can be smoothed in two ways as

$$X \cong E \# S_2, \quad X \cong S_1 \# E.$$

The basic idea of construction of f_1, f_2 is as follows:

$$X := S_1\#(\Sigma\#|E_8|)\#S_2,$$
$$f_1 := \varphi_1\#\,\mathrm{id}_{(\Sigma\#|E_8|)}\#\,\mathrm{id}_{S_2},$$
$$f_2 := \mathrm{id}_{S_1}\#\,\mathrm{id}_{(\Sigma\#|E_8|)}\#\varphi_2.$$

The precise definition is slightly complicated.

Lemma 1

f_1 is smoothable for $X \cong S_1\#E$.
f_2 is smoothable for $X \cong E\#S_2$.

By Proposition 2, at least one of f_1. f_2 should be nonsmoothable if

(1) $f_i^*c \cong c$ $(i = 1, 2)$, and
(2) $w_2\left(H^+_{(f_1,f_2)}\right) \neq 0$.

(1) can be easily seen as before.
For (2), by construction, $H^+_{(f_1,f_2)} \to S^1 \times S^1$ can be written as

$$H^+_{(f_1,f_2)} \cong p_1^*\eta \oplus p_2^*\eta,$$

where $\eta \to S^1$ is a nontrivial line bundle, and $p_i\colon S^1 \times S^1 \to S^1$ is the i-th projection.
Thus, $w_2\left(H^+_{(f_1,f_2)}\right) \neq 0$.

□

Seiberg-Witten moduli spaces for families

- X: a closed ori. smooth 4-manifold with $b_1 = 0$.
- c: a Spin^c-structure on X, $L = \det c$.
- g: a Riemannian metric.
- Fix a g-self-dual 2-form $\mu \in \Omega^{+_g}(X)$.

The Seiberg-Witten equations for the parameter (g, μ)

$$(SW) \begin{cases} D_A\psi = 0, \\ F_A^{+_g} = (\psi \otimes \psi^*)_0 + i\mu, \end{cases}$$

where

- A: $\mathrm{U}(1)$-connection on $L = \det c$,
- ψ: positive spinor.

The moduli space

$$\mathcal{M} = \mathcal{M}(X, c, g, \mu) = \{ \text{ solutions of (SW) } \}/\mathcal{G},$$

where $\mathcal{G} = \mathrm{Map}(X, S^1)$ is the gauge transformation group.

Properties

- $\mathcal{M}$ is compact.
- For a generic choice of (g, μ), $\mathcal{M}$ becomes a $d(c)$-dim. manifold except quotient singularities(=reducibles), where

$$d(c) = \frac{1}{4}(c_1(L)^2 - \mathrm{Sign}(X)) - (1 - b_1 + b^+).$$

- If X : Enriques & c : a torsion Spin^c-str. $\Rightarrow d(c) = 0$.

Proof of Proposition 1

- Let X : Enriques & c : a torsion Spin^c-str. $\Rightarrow d(c) = 0$.
- Suppose an ori. pres. diffeo $f\colon X \to X$ s.t. $f^*c \cong c$ given.
- Consider the mapping torus $X_f = (X \times [0,1])/f \to S^1$.
- A family of Spin^c-structure $c_f = (c \times [0,1])/f^*$.
- For $b \in S^1$, let (X_b, c_b) be the fibre of $(X_f, c_f) \to S^1$ over b.
- The bundle of parameters:

$$\begin{array}{l}\Pi := \{(g_b, \mu_b) \in Met(X_b) \times \Omega^2(X_b) \,\big|\, *_{g_b} \mu_b = \mu_b\} \\ \downarrow \\ S^1\end{array}$$

- Choose a section $\eta\colon S^1 \to \Pi$.

  $\Rightarrow$ A family of SW-eqn. on (X_f, c_f).

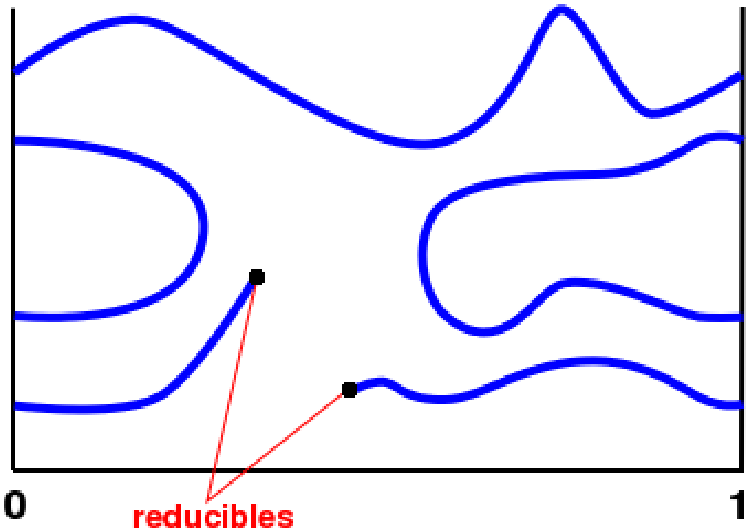
- The moduli space for the family (X_f, c_f):

$$\mathcal{M}(X_f, c_f, \eta) = \coprod_{b \in S^1} \mathcal{M}(X_b, c_b, g_b, \mu_b).$$

- For generic η, $\mathcal{M}(X_f, c_f, \eta)$ becomes a $(d(c)+1)$-dim. compact manifold outside reducibles.

- In our case, $d(c) = 0$,

$\Rightarrow$ $\mathcal{M}(X_f, c_f, \eta)$ is a cpt. 1-dim. manifold with boundaries = reducibles.



Note that the number of boundaries (= reducibles) is even.

Question Where do reducibles appear?

$\mathcal{M}(X_b, c_b, g_b, \mu_b)$ contains a reducible $\leftrightarrow$ A condition for (g_b, μ_b).

- Let us introduce an $\mathbb{R}$-line bundle $H_f^+ \to S^1$ by

$$H_f^+ = \coprod_{b \in S^1} H^{+g_b},$$

where H^{+g_b} is the space of g_b-self-dual harmonic 2-forms.

- Define the section $s_\eta \colon S^1 \to H_f^+$ by

$$s_\eta(b) := P^{+g_b}(2\pi c_1(L) - \mu_b),$$

where P^{+g_b} is the projection to g_b-self-dual harmonic part, and $c_1(L)$ is assumed as a harmonic 2-form.

Lemma

- $s_\eta(b) = 0 \Leftrightarrow \mathcal{M}(X_b, c_b, g_b, \mu_b)$ contains a reducible. In fact,

$$s_\eta^{-1}(0) \stackrel{1:1}{\longleftrightarrow} \{ \text{ reducibles } \}$$

- η: generic $\Rightarrow s_\eta \pitchfork$ (0-section).

Proof of Proposition 1

$$\#\{ \text{ boundaries } \} = \#\{ \text{ reducibles } \} = \# s_\eta^{-1}(0) \text{ is even}$$

$\Rightarrow$ H_f^+ is a trivial line bundle . $(w_1(H_f^+) = 0.)$

$\Rightarrow$ f preserves the ori. of $H^+(X)$.

□

Proof of Proposition 2

- Let $X := E\#(S^2 \times S^2)$, c: a torsion Spin^c-str. $\Rightarrow d(c) = -1$.
- Suppose two commutative ori. pres. diffeos $f_1, f_2 \colon X \to X$ s.t. $f_1^* c \cong f_2^* c \cong c$ given.
- Consider the "double" mapping torus

$$X_{(f_1,f_2)} = (X \times [0,1] \times [0,1])/(f_1, f_2) \to T^2.$$

Lemma

If $f_1^* c \cong f_2^* \cong c$, then $\exists$ a Spin^c-str. $\tilde{c}$ on $X_{(f_1,f_2)}$ s.t.

$$\tilde{c}|_{X_b} \cong c \quad \text{ for } \forall b \in T^2.$$

- The bundle of parameters:

$$\Pi := \{(g_b, \mu_b) \in Met(X_b) \times \Omega^2(X_b) \,|\, *_{g_b} \mu_b = \mu_b\} \to T^2.$$

- Choose a section $\eta\colon T^2 \to \Pi$.
- The moduli space for the family $(X_{(f_1,f_2)}, \tilde{c})$:

$$\mathcal{M}(X_{(f_1,f_2)}, \tilde{c}, \eta) = \coprod_{b \in T^2} \mathcal{M}(X_b, c_b, g_b, \mu_b).$$

- For generic η, $\mathcal{M}(X_{(f_1,f_2)}, \tilde{c}, \eta)$ becomes a $(d(c)+2)$-dim. compact manifold outside reducibles.
- In our case, $d(c) = -1$

$\Rightarrow$ $\mathcal{M}(X_{(f_1,f_2)}, \tilde{c}, \eta)$ is a cpt 1-dim. manifold with boundaries = reducibles.

- Define an $\mathbb{R}^2$-vector bundle $H^+_{(f_1,f_2)} \to T^2$ by

$$H^+_{(f_1,f_2)} = \coprod_{b \in T^2} H^{+g_b},$$

where H^{+g_b} is the space of g_b-self-dual harmonic 2-forms.

Lemma

- $\exists$ a section s_η of $H^+_{(f_1,f_2)} \to T^2$ s.t.

$$s_\eta^{-1}(0) \overset{1:1}{\longleftrightarrow} \{\text{ reducibles }\}$$

- η: generic $\Rightarrow s_\eta \pitchfork$ (0-section).

Proof of Proposition 2

$$\#\{\text{ boundaries }\} = \#\{\text{ reducibles }\} = \# s_\eta^{-1}(0) \text{ is even}$$
$$\Rightarrow \quad w_2(H^+_{(f_1,f_2)}) = 0.$$

□

The statement of results
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Proof of Proposition 1
Proof of Proposition 2

Final remark

- For an ori. closed smooth X^4 with intersection form I_X,

$$\begin{aligned}
\mathrm{Diff}^+(X) &:= \{\text{ orientation preserving diffeomorphisms }\},\\
\mathrm{Homeo}^+(X) &:= \{\text{ orientation preserving homeomorphisms }\},\\
O = O(H_2) &:= \{\text{ automorphisms of } H_2(X;\mathbb{Z}) \text{ preserving } I_X \}.
\end{aligned}$$

- We have homomorphisms

$$\begin{aligned}
\psi &\colon \mathrm{Diff}^+(X) \to O,\\
\phi &\colon \mathrm{Homeo}^+(X) \to O.
\end{aligned}$$

Problem Determine $\operatorname{im}\psi$ and $\operatorname{im}\phi$.

Nobuhiro Nakamura
Smoothability of $\mathbb{Z} \times \mathbb{Z}$-actions on 4-manifolds

For $X = K3$,
[Matumoto '85] $\operatorname{im}\psi = O'$, where

$$O' = \{ \text{ automorphisms of } (H_2, I_X) \text{ preserving the ori. of } H^+ \}.$$

O' is an index-2 subgroup of O.

[Freedman] $\operatorname{im}\phi = O$.

For $X =$ Enriques,
[Lönne '98] $\operatorname{im}\psi = \operatorname{im}\phi = O$.

By Proposition 1 & Lönne's argument, we can prove the following:

For a Spin^c-str. c on X,

$$\begin{aligned} \mathrm{Diff}^+(X,c) &:= \{ \text{ ori. pres. diffeomorphisms preserving } c \}, \\ \mathrm{Homeo}^+(X,c) &:= \{ \text{ ori. pres. homeomorphisms preserving } c \}, \\ \psi_c &\colon \mathrm{Diff}^+(X,c) \to O, \\ \phi_c &\colon \mathrm{Homeo}^+(X,c) \to O. \end{aligned}$$

Proposition

Let X be an Enriques surf. and c a torsion Spin^c-structure.
Then $\operatorname{im}\psi_c = O'$ and $\operatorname{im}\phi_c = O$.