Mod 2 Seiberg-Witten simple type

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X : closed connected oriented smooth 4-manifold $b^+ \geq 2$

always assumed

$\mathfrak{s}$: Spin^c structure on X $K := c_1(\mathfrak{s})$

SW-moduliの次元

$$d(\mathfrak{s}) := d(K) := \frac{1}{4}\left(K^2 - 2e(X) - 3\sigma(X)\right)$$

Euler signature

Simple type conjecture

$$\boxed{d(\mathfrak{s}) > 0 \Rightarrow SW_X(\mathfrak{s}) = 0}$$

SW simple type

$$SW_X(\mathfrak{s}) := \langle U^{\frac{d(\mathfrak{s})}{2}}, [\mathcal{M}] \rangle \in \mathbb{Z}$$

SW-moduli

$\mathcal{M} \subset \mathcal{B}^* \simeq \mathbb{C}P^\infty \times T^{b_1(X)}$

U: generator of $H^2(\mathbb{C}P^\infty)$

$\boxed{\text{Main Theorem}}$ (Kato - N. - Yasui 2020)

$X : b_2^+ - b_1 > 1 \quad b_2^+ - b_1 \underset{(4)}{\equiv} 3$

Note
$b^+ - b_1$: even $\Rightarrow d(\mathfrak{s})$: odd $\Rightarrow SW_X(\mathfrak{s}) = 0$

$\{\delta_1, \cdots, \delta_k\}$: a generating set of $H^1(X;\mathbb{Z})$

Suppose $\forall i,j \quad \delta_i \cup \delta_j$ is a torsion or divisible by 2.

$$d(\mathfrak{s}) > 0 \Rightarrow SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 0$$

$\leftarrow$ Mod 2 SW simple type

<u>Cor</u> $b_2^+ > 1 \quad \boxed{b_1 \leq 1}$

$b_2^+ - b_1 \underset{(4)}{\equiv} 3$

$$d(\mathfrak{s}) > 0 \Rightarrow SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 0$$

<u>Remark</u> The case of $b_1 = 0$ is proved by [Furuta]

Applications

① Adjunction inequalities

$\boxed{\text{Th'm}}$ (KNY) X : as in Main Th'm

$S^2 \subset X$ immersed sphere with $\begin{cases} p_+ & \text{positive double points} \\ p_- & \text{negative} \quad " \end{cases}$

$\alpha := P.D.[S^2] \in H^2(X;\mathbb{Z})$

$p_+ > 0$ $\quad \alpha\cdot\alpha < 0$ $\quad SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1 \quad \Rightarrow \quad |K\cdot\alpha| + \alpha\cdot\alpha \leq 2p_+ - 2$

$\boxed{\text{Th'm}}$ (KNY) $X : b^+ > 1 \quad b^+ - b_1 \underset{(4)}{\equiv} 3 \quad b_1 \leq 1$

$\Sigma \subset X$ embedded surface w/ genus $= g$

$\alpha := P.D[\Sigma] \in H^2(X;\mathbb{Z})$

$g > 0$ $\quad \alpha\cdot\alpha < 0$ $\quad SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1 \quad \Rightarrow \quad |K\cdot\alpha| + \alpha\cdot\alpha \leq 2g - 2$

② **Conjecture** (For Donaldson inv., Kotschick)

X : $b^+ > 1$ $b^- > 1$ $\pi_1 X = 1$

$\bar{X}$: X with reversed orientation

$$SW_X \equiv 0 \quad \text{or} \quad SW_{\bar{X}} \equiv 0$$

Def geometrically simply connected $\underset{\text{def}}{\Longleftrightarrow}$ $\exists$ handle decomposition without 1-handles

$\boxed{\text{Thm}}$ (KNY) X: geom. simply connected

$$b_2^+ \underset{(4)}{\not\equiv} 1 \qquad b_2^- \underset{(4)}{\not\equiv} 1$$

$$\Rightarrow SW_X \underset{(2)}{\equiv} 0 \quad \text{or} \quad SW_{\bar{X}} \underset{(2)}{\equiv} 0$$

NO.

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Contents

Background (simple type)

Kronheimer-Mrowka : Structure theorem of Donaldson invariants.

X : $b_2^+ > 1$ $b_2^+ - b_1$: odd

$$\mathbb{A}(X) := \mathrm{Sym}(H_{even}(X;\mathbb{R})) \otimes \Lambda H_{odd}(X;\mathbb{R})$$

Donaldson invariant

$$D_X^w : \mathbb{A}(X) \to \mathbb{R}$$ $w = \det(E)$

KM simple type $D_X^w(x^2 z) = 4 D_X^w(z)$ $z \in \mathbb{A}(X)$

x : the generator of $H_0(X)$

[Kronheimer - Mrowka 1995] X : $b_2^+>1$ b_2^+ : odd $b_1=0$ $\boxed{\text{KM simple type}}$

① $\exists$ basic classes $K_1, \dots, K_r \in H^2(X;\mathbb{Z})$

$\exists\ \beta_1, \dots, \beta_r \in \mathbb{Q}$

D_X^w is determined by $K_1, \dots, K_r, \beta_1, \dots, \beta_r$

② $\Sigma \subset X$ embedded surface $[\Sigma]^2 \geq 0$ $[\Sigma]$ non-torsion

$$2g(\Sigma)-2 \geq [\Sigma]^2 + K_r \cdot [\Sigma]$$

If $\exists\ \Sigma \subset X$ s.t. $g(\Sigma) \geq 2$

$\boxed{2g(\Sigma)-2 = [\Sigma]^2}$

tight

$\Longrightarrow$ X : KM simple type

SW simple type も出る

KM Conjecture $K_r^2 = 2e(X) + 3\sigma(X)$ $\Leftrightarrow d(K_r)=0$

KM Conjecture $K_X^2 = 2e(X) + 3\sigma(X) \Leftrightarrow d(K_X) = 0$

Witten's conjecture (1994) $\beta_i = SW_X(k_i)$

If Witten conj. true $\Rightarrow$ KM conj means SW simple type conj.
when $b_1 = 0$.

Feehan-Leness (2015)

X: $b_2^+ > 1$ b_2^+ : odd $b_1 = 0$ $\boxed{\text{SW-simple type}}$ + Some topological conditions

$\Rightarrow$ X : KM simple type & Witten conj. is true.

SW simple type

◎ Non-simple type 4-manifold $\boxed{\text{Not Found}}$

- Symplectic
- $\exists$ tight surface $\Sigma \subset X$
  $2g(\Sigma) - 2 = [\Sigma]^2 > 0$

$\Longrightarrow$ SW simple type

Kronheimer-Mrowka: Monopoles and three-manifolds
It is very unclear to what extent the familiar examples of 4-manifolds are representative of the general case, so there is little reason to extrapolate with any confidence.

[Furuta 1998] $X : b_2^+ > 1 \quad b_2^+ : \text{odd} \quad b_1 = 0$

$\mathfrak{s} :$ Spin^c str. on $X \quad d(\mathfrak{s}) > 0$

$$\left(\frac{\log(1+x)}{x}\right)^{\frac{b^+-1}{2}} = \left(1 + \frac{x}{2} + \frac{x^2}{3} + \cdots\right)^{\frac{b^+-1}{2}} = 1 + a_1 x + a_2 x^2 + \cdots$$ と展開

$$1 \leq {}^\forall i \leq \frac{d(\mathfrak{s})}{2} \qquad a_i SW_X(\mathfrak{s}) \in \mathbb{Z}$$

← Proof

Compare the ordinary SW & k-version of SW

Cor $b_2^+ \equiv 3 \ (4)$, $b_1 = 0 \Rightarrow SW_X(\mathfrak{s}) \equiv 0 \ (2)$

ちなみに $b_2^+ \equiv 1 \ (4) \Rightarrow a_1 \in \mathbb{Z}$

Proof of $\boxed{\text{Main Thm}}$

$\boxed{\text{Main Thm}}$ $X : b_2^+ - b_1 > 1 \quad b_2^+ - b_1 \underset{(4)}{\equiv} 3$

$\{\delta_1, \dots, \delta_k\}$: a generating set of $H^1(X;\mathbb{Z})$

$\forall i,j \quad \delta_i \cup \delta_j$ is a torsion or divisible by 2

$$d(\mathfrak{s}) > 0 \implies SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 0$$

次を示せば十分

$X : b_2^+ - b_1 > 1 \quad b_2^+ - b_1 \underset{(4)}{\equiv} 3$

$$\left.\begin{array}{l} SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1 \\ \forall i,j \ \langle c_1(\mathfrak{s}) \cup \delta_i \cup \delta_j, [X] \rangle \underset{(4)}{\equiv} 0 \end{array}\right\} \implies d(\mathfrak{s}) = 0$$

Proof Suppose $d(\mathfrak{s}) = 2n > 0$ $K := c_1(\mathfrak{s})$

[Fintushel - Stern (1995)] (Blow up formula)

$d(\mathfrak{s}) - r(r+1) \geq 0$ $d(K \pm (2r+1)E) = d(K) - r(r+1)$

$\Longrightarrow SW_X(K) = SW_{X \# \overline{\mathbb{CP}^2}}(K \pm (2r+1)E)$

↑ P.D. of the exceptional sphere

$X_n := X \# n\overline{\mathbb{CP}^2}$ $K_n := K + 3E_1 + \cdots + 3E_n$

$\exists \mathfrak{s}_n$ s.t. $c_1(\mathfrak{s}_n) = K_n$ $SW_{X_n}(\mathfrak{s}_n) = SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1$ $\underline{d(\mathfrak{s}_n) = 0}$

$(X_n, \mathfrak{s}_n)$ satisfies $b_2^+ - b_1 > 1$, $b_2^+ - b_1 \underset{(4)}{\equiv} 3$, $\forall i,j$ $\langle c_1(\mathfrak{s}_n) \cup \delta_i \cup \delta_j, [X] \rangle \underset{(4)}{\equiv} 0$, $d(\mathfrak{s}_n) = 0$, $SW_{X_n}(\mathfrak{s}_n) \underset{(2)}{\equiv} 1$

$(K3, \mathfrak{s}_0)$ $Z := X_n \# K3$

↳ canonical Spin^c

By [Ishida-Sasahira (2015)]

$$SW_Z^{Spin}(\mathfrak{s}_n \# \mathfrak{s}_0) \neq 0 \text{ in } \Omega_1^{Spin} \cong \mathbb{Z}/2$$

↑ Spin cobordism SW invariant

By the adjunction inequality,

$\Sigma \subset Z$ embedded surface $\quad \alpha = P.D[\Sigma]$ non-torsion

$\alpha \cdot \alpha \geq 0$

$$\Rightarrow \quad 2g(\Sigma) - 2 \geq |K_n \cdot \alpha| + \alpha \cdot \alpha$$

[Hamilton (2014)] $\exists\, \Sigma_0 \subset K3$ s.t. $g(\Sigma_0) \geq 2$ $\quad \underbrace{2g(\Sigma_0) - 2 = \alpha_0 \cdot \alpha_0}_{\text{tight}}$ $\quad \alpha_0 := P.D.[\Sigma_0]$

S_1 : the exceptional sphere $\quad P.D.[S_1] = -E_1$

$\Sigma := \Sigma_0 \# S_1 \subset Z \quad \Rightarrow \quad \cdot\, g(\Sigma) = g(\Sigma_0) \geq 2$

$\qquad\qquad\qquad\qquad\qquad\quad \cdot\, P.D.[\Sigma] = \alpha_0 - E_1$

$$2g(\Sigma) - 2 \geq \underbrace{|K_n \cdot (\alpha_0 - E_1)|}_{= |3E_1 \cdot (-E_1)| = 3} + \underbrace{(\alpha_0 - E_1)\cdot(\alpha_0 - E_1)}_{= \alpha_0^2 - 1,\ \alpha_0^2 = 2g-2} = 2g(\Sigma) \quad \text{矛盾}$$

$\therefore\ d(s) = 0$ ▨

Adjunction inequality

[Fintushel - Stern (1995)] (Adjunction inequality for immersed spheres)

$X : b_2^+ > 1$ $\quad \mathfrak{s}$: Spin^c str. s.t. $SW_X(\mathfrak{s}) \neq 0$ $\quad K = c_1(\mathfrak{s})$

$S^2 \subset X$ immersed sphere with p_+ positive double points

$\alpha = P.D.[S^2]$: non-torsion

$\Rightarrow$ Either $\quad 2p_+ - 2 \geq |K \cdot \alpha| + \alpha \cdot \alpha$

or

$SW_X(\mathfrak{s}) = SW_X(\mathfrak{s} + \varepsilon\alpha)$ $\quad \varepsilon = \pm 1$ sign of $K \cdot \alpha$

Our case $\quad X$: as in $\boxed{\text{Main Thm}}$ $\Rightarrow SW_X(\mathfrak{s}') \underset{(2)}{\equiv} 0$ if $d(\mathfrak{s}') > 0$

Suppose $SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1$ & $2p_+ - 2 < |K \cdot \alpha| + \alpha \cdot \alpha$

$$\Rightarrow \begin{cases} SW_X(\mathfrak{s} + \varepsilon\alpha) = SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1 \\ d(\mathfrak{s} + \varepsilon\alpha) = d(\mathfrak{s}) + |K \cdot \alpha| + \alpha \cdot \alpha > 0 \end{cases}$$

$\to$ 矛盾

Geometrically simply connected 4-manifold & 2-handle neighborhood

Def $\alpha \in H_2(X;\mathbb{Z})$ is represented by a 2-handle neighborhood

$\underset{\text{def}}{\Leftrightarrow}$ $\exists\, W \subset X$ codim 0 submanifold s.t. $\cdot\ W = \text{4-ball} \cup \text{2-handle}$

$$\cdot\ \begin{array}{ccc} H_2(W;\mathbb{Z}) = \mathbb{Z} & \longrightarrow & H_2(X;\mathbb{Z}) \\ \Psi & & \Psi \\ \text{generator} & \longmapsto & \alpha \end{array}$$

[Yasui; 2019]

X: geom. simply connected $\Rightarrow$ $\forall \alpha \in H_2(X;\mathbb{Z})$ is represented by a 2-handle nbd.

$\boxed{\text{Thm}}$ X: $b_2^+ - b_1 > 1$, $b_2^+ - b_1 \not\equiv_{(4)} 1$; $b_2^- - b_1 > 1$, $b_2^- - b_1 \not\equiv_{(4)} 1$; $\forall i,j\ \ d_i \cup d_j$: torsion or divisible by 2

$\exists \alpha \in H_2(X;\mathbb{Z})$: non-torsion represented by a 2-handle nbd $\Big\} \Rightarrow SW_X \equiv_{(2)} 0$ or $SW_{\overline{X}} \equiv_{(2)} 0$

<u>Proof</u> By $\boxed{\text{Main Thm}}$ $SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 0$ if $d(\mathfrak{s}) > 0$

$SW_{\bar{X}}(\mathfrak{s}') \underset{(2)}{\equiv} 0$ if $d(\mathfrak{s}') > 0$

Suppose $\exists \mathfrak{s}$ on X s.t. $SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1$ $d(\mathfrak{s}) = 0$

$\exists \mathfrak{s}'$ on $\bar{X}$ s.t. $SW_{\bar{X}}(\mathfrak{s}') \underset{(2)}{\equiv} 1$ $d(\mathfrak{s}') = 0$

By [Ishida-Sasahira] $SW^{Spin}_{X \# \bar{X}}(\mathfrak{s} \# \mathfrak{s}') \neq 0$ in $\Omega_1^{Spin} \cong \mathbb{Z}/2$

<u>Lemma 1</u> (Yasui) $H_2(X;\mathbb{Z}) = H_2(\bar{X};\mathbb{Z})$

$\alpha \longleftrightarrow \bar{\alpha}$

$\alpha - \bar{\alpha} \in H_2(X \# \bar{X};\mathbb{Z})$ is represented by an embedded sphere S with $[S]^2 = 0$

<u>Lemma 2</u> $\left.\begin{array}{l} \exists\, S^2 \subset Z^4 \text{ embedded} \text{ s.t. } [S^2] \text{ non-torsion} \\ [S^2]^2 = 0 \end{array}\right) \Rightarrow SW^{Spin}_Z(\mathfrak{s}) = 0 \quad \forall \mathfrak{s}$

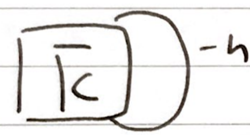
矛盾 $\boxtimes$

Lemma 1 (Yasui) $\alpha \in H_2(X;\mathbb{Z})$ non-torsion represented by a 2-handle nbd W.

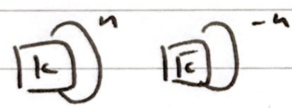
$\Rightarrow$ $\alpha - \bar{\alpha} \in H_2(X \# \bar{X};\mathbb{Z})$ is represented by an embedded 2-sphere S with $[S]^2 = 0$.

Proof

$\alpha \leftrightarrow W =$ 

$-\bar{\alpha} \leftrightarrow \bar{W} =$ 

$\bar{K}$: mirror of K

$W \sqcup \bar{W} =$ 

K の 2-handle を $\bar{K}$ に slide $\rightarrow$ 0-framed $\underbrace{K \# \bar{K}}_{\text{slice}}$

$\Rightarrow$ $\exists\, S^2$ represents $\alpha - \bar{\alpha}$ $[S^2]^2 = 0$ ∎