Upper bounds for virtual dimensions of Seiberg–Witten moduli spaces

Nobuhiro Nakamura (Fukushima Medical University)

2023/3/24

- X : a closed connected oriented smooth 4-manifold with $b_2^+(X) \geq 2$

always assumed

- $\mathfrak{s}$: spin-c structure on X
- $d(\mathfrak{s}) = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - \sigma(X)\right) - (1 - b_1 + b_2^+)$ ←The virtual dim. of the SW moduli space

Simple type conjecture $SW_X(\mathfrak{s}) \neq 0 \Rightarrow d(\mathfrak{s}) = 0$

- $SW_X(\mathfrak{s}) = \left\langle U^{\frac{d(\mathfrak{s})}{2}}, [\mathscr{M}] \right\rangle \in \mathbb{Z}$ if $d(\mathfrak{s})$ is even. $SW_X(\mathfrak{s}) = 0$ if $d(\mathfrak{s})$ is odd.
- $\mathscr{M}$: the Seiberg-Witten moduli space $\mathscr{M} \subset \mathscr{B}^* \simeq \mathbb{C}P^\infty \times T^{b_1(X)}$
- U : a canonical generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$

▸ Simple type conjecture is still open.

Mod 2 simple type

Theorem (Kato-N.-Yasui, 2020)

- $b_2^+ - b_1 > 1,\ b_2^+ - b_1 \not\equiv 3 \mod 4$ $\quad b_2^+ - b_1$: odd $\Leftrightarrow d(\mathfrak{s})$: even

- $\{\delta_1, \ldots, \delta_k\}$ a generating set of $H^1(X;\mathbb{Z})$

  Suppose $\forall i, j\ \delta_i \cup \delta_j$ is a torsion or divisible by 2

$$SW_X(\mathfrak{s}) \not\equiv 0 \mod 2 \Rightarrow d(\mathfrak{s}) = 0$$

Theorem (Baraglia, 2023)

- $b_2^+ > 0$ & $\mathfrak{s}$: spin $\quad SW_X(\mathfrak{s}) \not\equiv 0 \mod 2 \Rightarrow d(\mathfrak{s}) = 0$

What about mod m simple type for $m > 2$?

- We do not have a result like

$$SW_X(\mathfrak{s}) \not\equiv 0 \mod m \Rightarrow d(\mathfrak{s}) = 0$$

- But we obtain a result as

$$SW_X(\mathfrak{s}) \not\equiv 0 \mod m \Rightarrow \exists \text{upper bound on } d(\mathfrak{s})$$

Main Theorem (Kato-Kishimoto-N.-Yasui, 2023) arXiv:2111.15201v2

p : prime, $b_2^+ \geq 2$, $b_1 = 0$. For $r \in \mathbb{Z}$ s.t. $1 \leq r < p(p-1)$

$$SW_X(\mathfrak{s}) \not\equiv 0 \mod p^r \Rightarrow d(\mathfrak{s}) \leq 2r(p-1) - 2$$

if the following conditions are satisfied

1. $k := \frac{b_2^+ - 1}{2} \not\equiv 0, 1, \ldots, r_p \mod p$, where $r_p \in \mathbb{Z}$ s.t. $\begin{cases} 1 \leq r_p \leq p-1 & \\ r_p = 1 & \text{if } r \equiv 0 \mod p \\ r_p \equiv r \mod p & \text{if } r \not\equiv 0 \mod p \end{cases}$

2. For t such that $k - t + 2 \equiv 0 \mod p$ and $3 \leq t \leq r$, let $a(k,t) := \frac{k-t+2}{p}$.

$$\begin{aligned} &3a(k,t) + 5 \not\equiv 0 && \mod p && (t \equiv 0 \mod p \text{ and } t \geq p > 3) \\ &a(k,t) + 2 \not\equiv 0 && \mod p && (t \equiv 1 \mod p \text{ and } t > p) \\ &3a(k,t) + 4 \not\equiv 0 && \mod p && (t \equiv 3 \mod p) \\ &(2t-3)a(k,t) + 3t - 5 \not\equiv 0 && \mod p && (t \equiv 4, 5, \cdots, p-1 \mod p). \end{aligned}$$

Main Theorem (Kato-Kishimoto-N.-Yasui, 2023) arXiv:2111.15201v2

p : prime, $b_2^+ \geq 2$, $b_1 = 0$. For $r \in \mathbb{Z}$ s.t. $1 \leq r < p(p-1)$

$$SW_X(\mathfrak{s}) \not\equiv 0 \mod p^r \Rightarrow d(\mathfrak{s}) \leq 2r(p-1) - 2$$

if the following conditions are satisfied

1. $k := \frac{b_2^+ - 1}{2} \not\equiv 0, 1, \ldots, r_p \mod p$, where $r_p \in \mathbb{Z}$ s.t. $\begin{cases} 1 \leq r_p \leq p-1 & \\ r_p = 1 & \text{if } r \equiv 0 \mod p \\ r_p \equiv r \mod p & \text{if } r \not\equiv 0 \mod p \end{cases}$
2. For t such that $k - t + 2 \equiv 0 \mod p$ and $3 \leq t \leq r$, let $a(k,t) := \frac{k-t+2}{p}$.

$$\begin{array}{lll} 3a(k,t) + 5 \not\equiv 0 & \mod p & (t \equiv 0 \\ a(k,t) + 2 \not\equiv 0 & \mod p & (t \equiv 1 \\ 3a(k,t) + 4 \not\equiv 0 & \mod p & (t \equiv 3 \\ (2t-3)a(k,t) + 3t - 5 \not\equiv 0 & \mod p & (t \equiv 4, \end{array}$$

These conditions are used for the computations of the attaching maps of $\mathbb{C}P^{\bullet}_{(p)}$

Corollary $(r = 1)$

p : prime, $b_2^+ \geq 2$ & b_2^+ odd, $b_1 = 0$. $\frac{b_2^+ - 1}{2} \not\equiv 0 \mod p$

$$SW_X(\mathfrak{s}) \not\equiv 0 \mod p \Rightarrow d(\mathfrak{s}) \leq 2p - 4$$

- $p = 2 \Rightarrow d(\mathfrak{s}) = 0$ ← Corollary recovers [Kato-N.-Yasui] in the case when $b_1 = 0$
- $p = 3 \Rightarrow d(\mathfrak{s}) = 0,2$
- $p = 5 \Rightarrow d(\mathfrak{s}) = 0,2,4,6$

- This is the main theorem of arXiv:2111.15201v1

Upper bounds from [Bauer-Furuta]

Define $a_i^{(k)}$ by

$$\left(-\frac{\log(1-x)}{x}\right)^k = \left(1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^{n-1}}{n} + \cdots\right)^k = 1 + \sum_{i \geq 1} a_i^{(k)} x^i .$$

Theorem (Bauer-Furuta, 2004)

Let $k := \frac{b_2^+ - 1}{2}$. If $b_2^+ \geq 2$ & $b_1 = 0$,

then $SW_X(\mathfrak{s})$ is divisible by the denominator of $a_i^{(k)}$ for $1 \leq i \leq d(\mathfrak{s})/2$

- Upper bounds of $d(\mathfrak{s})$ can be obtained from this theorem.
- Corollary can be also deduced from this theorem.

Comparison between Main Theorem and [Bauer-Furuta]

$k = 1$ $(b_2^+ = 3)$ & $r = p$. Suppose $SW_X(\mathfrak{s}) \not\equiv 0 \mod p^r = p^p$

- [Bauer-Furuta] $\Rightarrow d(\mathfrak{s}) \leq 2p^p - 4$
- Main Theorem $\Rightarrow d(\mathfrak{s}) \leq 2p(p-1) - 2$

- We believe that all upper bounds from [Bauer-Furuta] can be proved by Main Theorem.
- We did not succeed in verifying it.

  One reason: the bounds from [Bauer-Furuta] are inexplicit
  - $a_i^{(k)}$ can be computed from the Bernoulli numbers.
  - No general formula to compute $a_i^{(k)}$ directly from k, i is known.

Contents

- Review on SW invariants & Bauer-Furuta's refinement
- Outline of the proof of Main Theorem (Main theorem → Key Lemma)
- p-localization & Toda bracket
- Outline of the proof of Key Lemma

Monopole map

$\mathfrak{s}$: spin-c structure. $S^{\pm}$: spinor bundles

Choose a reference $\mathrm{U}(1)$ connection A_0 on $L = \det S^{\pm}$

$$\mu : \sqrt{-1}\Omega^1(X) \times \Gamma(S^+) \to \sqrt{-1}\Omega^+(X) \times \Gamma(S^-) \qquad \mu(a, \phi) = \left(F^+_{A_0} + d^+a - (\phi \otimes \phi^*), D_{A_0+a}\phi\right)$$

- The Seiberg-Witten equations $\Leftrightarrow \mu = 0$
- μ is $\mathscr{G} = \mathrm{Map}(X, \mathrm{U}(1))$ equivariant.
- The SW moduli space $\mathscr{M} = \mu^{-1}(0)/\mathscr{G}$ $\leftarrow$ compact
- Perturb μ

$\phi \equiv 0$

$\Rightarrow$ $\mathscr{M}$ is a $d(\mathfrak{s})$-dim. compact manifold and contains no reducibles (if $b_2^+ \geq 1$)

Seiberg-Witten invariant

$$\mathscr{M} \subset \mathscr{B}^* = \frac{\sqrt{-1}\Omega^1(X) \times \left(\Gamma(S^+) \backslash \{0\}\right)}{\mathscr{G}} \simeq \mathbb{C}P^\infty \times T^{b_1}$$

$$SW_X(\mathfrak{s}) = \begin{cases} \left\langle U^{\frac{d(\mathfrak{s})}{2}}, [\mathscr{M}] \right\rangle & \text{if } d(\mathfrak{s}) \text{ even} \\ 0 & \text{if } d(\mathfrak{s}) \text{ odd} \end{cases}$$

U : a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$

Finite dimensional approximation

Assume $b_1 = 0$

$$m = \ell + c : \sqrt{-1}\ker d^* \times \Gamma(S^+) \to \sqrt{-1}\Omega^+(X) \times \Gamma(S^-) \quad \mathrm{U}(1) \text{ equivariant}$$

Linear part $\ell(a,\phi) = \left(d^+ a, D_{A_0}\phi\right)$ Quadratic part $c(a,\phi) = \left(F^+_{A_0} - (\phi \otimes \phi^*)_0, a \cdot \phi\right)$

- m is the restriction of μ to the global slice of the $\mathscr{G}$-action $\Rightarrow \mathscr{M} = m^{-1}(0)/\mathrm{U}(1)$

- Choose a finite dim. subspace $F \subset \sqrt{-1}\Omega^+(X) \times \Gamma(S^-) =: \mathscr{U}$ s.t. $\mathrm{Im}\,\ell + F = \mathscr{U}$
- Let $E := \ell^{-1}(F)$ & $p : \mathscr{U} \to F$ the projection
- <u>Finite dim. approximation</u> $f = \ell + pc : E \to F$ $\leftarrow$ $\mathrm{U}(1)$-equiv. proper

  $\Rightarrow f^{-1}(0)/\mathrm{U}(1)$ is an approximation of $\mathscr{M}$ for a large F

- Perturb f $\Rightarrow$ $f^{-1}(0)/\mathrm{U}(1)$ is a compact manifold cobordant to $\mathscr{M}$

SW inv is computed from finite dim. approx.

We assume $b_1 = 0$

Decompose $E = \mathbb{R}^y \oplus \mathbb{C}^z \subset \ker d^* \times \Gamma(S^+)$

$$M := f^{-1}(0)/\mathrm{U}(1) \subset \left(\frac{\mathbb{C}^z \backslash \{0\}}{\mathrm{U}(1)}\right) \times \mathbb{R}^y \simeq \mathbb{C}P^{z-1}$$

$$SW_X(\mathfrak{s}) = \begin{cases} \left\langle U^{\frac{d(\mathfrak{s})}{2}}, [M] \right\rangle & \text{if } d(\mathfrak{s}) \text{ even} \\ 0 & \text{if } d(\mathfrak{s}) \text{ odd} \end{cases}$$

Bauer-Furuta's stable cohomotopy refinement of $SW_X(\mathfrak{s})$

- f has a form, $f = \ell + pc\colon E = \mathbb{C}^{a+x} \oplus \mathbb{R}^y \to F = \mathbb{C}^x \oplus \mathbb{R}^{b+y}$ where $a = \mathrm{ind}_{\mathbb{C}} D_{A_0}$, $\quad b = b_2^+$

Recall $\ell = (d^+, D_{A_0})$

- Choose $R \gg 0$ s.t. $f^{-1}(0) \subset \mathring{B}_R(\mathbb{C}^{a+x} \oplus \mathbb{R}^y)$ ← open ball with radius R centered at 0
- Let $S_R(\mathbb{C}^{a+x} \oplus \mathbb{R}^y) = \partial B_R(\mathbb{C}^{a+x} \oplus \mathbb{R}^y)$

$$g_F\colon S_R(\mathbb{C}^{a+x} \oplus \mathbb{R}^y) \xrightarrow{f|_{S_R(\mathbb{C}^{a+x}\oplus\mathbb{R}^y)}} \left(\mathbb{C}^x \times \mathbb{R}^{b+y}\right) \backslash \{0\} \xrightarrow{\simeq} S(\mathbb{C}^x \oplus \mathbb{R}^{y+b})$$

- If $F' = F \oplus U \Rightarrow g_{F'} \simeq g_F * \mathrm{id}_U$, where '$*$' is join. Assume $b = b_2^+ \geq 2$

$$\widetilde{SW}_X(\mathfrak{s}) := [g_F] \in \{S(\mathbb{C}^a), S(\mathbb{R}^b)\}^{\mathrm{U}(1)} = \operatorname*{colim}_{x,y\to\infty} \left[S(\mathbb{C}^{a+x} \oplus \mathbb{R}^y), S(\mathbb{C}^x \oplus \mathbb{R}^{b+y})\right]^{\mathrm{U}(1)}$$

$$= \operatorname*{colim}_{y\to\infty} \left[\mathbb{C}P^{a-1} * S^{y-1}, S^{b-1} * S^{y-1}\right] = \pi^{b-1}\left(\mathbb{C}P^{a-1}\right)$$

Relation between $SW_X(\mathfrak{s})$ and $\widetilde{SW}_X(\mathfrak{s})$

Cohomotopy Hurewicz map

$$\mathrm{hur}\colon \pi^{b-1}\left(\mathbb{C}P^{a-1}\right) \to H^{b-1}(\mathbb{C}P^{a-1}),$$

$$\left[\alpha\colon \mathbb{C}P^{a-1} \to S^{b-1}\right] \mapsto \alpha^*\eta,$$

where $\eta \in H^{b-1}(S^{b-1})$ generator

Theorem([Bauer-Furuta]) $\mathrm{hur}\left(\widetilde{SW}_X(\mathfrak{s})\right) = SW_X(\mathfrak{s}) \cdot U^k, \quad k = \dfrac{b-1}{2}$

- $d = d(\mathfrak{s}) = 2a - b - 1 = 2(a-1) - 2k, \quad \widetilde{SW}_X(\mathfrak{s}) \in \pi^{2k}\left(\mathbb{C}P^{k+d/2}\right)$
- If $d = 0 \Rightarrow \mathrm{hur}\colon \pi^{2k}\left(\mathbb{C}P^k\right) = \mathbb{Z} \overset{\cong}{\to} H^{2k}(\mathbb{C}P^k) = \mathbb{Z}$
- $\widetilde{SW}_X(\mathfrak{s})$ is stronger than $SW_X(\mathfrak{s})$. $\exists(X, \mathfrak{s})$ s.t. $\widetilde{SW}_X(\mathfrak{s}) \neq 0$ but $\mathrm{hur}\left(\widetilde{SW}_X(\mathfrak{s})\right) = 0$

[Bauer] $K3\#K3$ etc.

Methods of the proofs: Main Theorem and [Bauer-Furuta]

- Both of the proofs use $\mathrm{hur}\left(\widetilde{SW}_X(\mathfrak{s})\right) = SW_X(\mathfrak{s}) \cdot U^k$
- [Bauer-Furuta] uses the K-version of the Hurewicz map too, and compares the ordinary SW inv. with the K-version of SW inv.
- In the proof of Main Theorem, the cohomotopy groups are more directly analyzed by using techniques in homotopy theory such as p-localization and Toda bracket.

Outline of the proof of Main Theorem

- To prove: $p^r \nmid SW_X(\mathfrak{s}) \Rightarrow d \leq 2r(p-1) - 2$

- Suppose $d \geq 2r(p-1)$. Then prove $p^r \mid SW_X(\mathfrak{s})$

$$\begin{array}{ccc}
\widetilde{SW}_X(\mathfrak{s}) \in \pi^{2k}(\mathbb{C}P^{k+d/2}) & \xrightarrow{\text{hur}} & H^{2k}(\mathbb{C}P^{k+d/2}) \cong \mathbb{Z} \ni SW_X(\mathfrak{s}) \\
\downarrow i_2^* & & \cong \downarrow i_2^* \\
\pi^{2k}(\mathbb{C}P^{k+r(p-1)}) & \xrightarrow{\text{hur}} & H^{2k}(\mathbb{C}P^{k+r(p-1)}) \cong \mathbb{Z} \\
\downarrow i_1^* & & \cong \downarrow i_1^* \\
\pi^{2k}(\mathbb{C}P^{k}) & \xrightarrow[\cong]{\text{hur}} & H^{2k}(\mathbb{C}P^{k}) \cong \mathbb{Z}
\end{array}$$

$$\mathbb{C}P^k \underset{i_1}{\hookrightarrow} \mathbb{C}P^{k+r(p-1)} \underset{i_2}{\hookrightarrow} \mathbb{C}P^{k+d/2}$$

Key lemma

We will use p-localization.

$$\mathbb{Z}_{(p)} = p\text{-localization of } \mathbb{Z} \text{ at } p = \left\{ \frac{b}{a} \,\middle|\, a, b\colon \text{ coprime, } p \nmid a \right\} \subset \mathbb{Q}$$

Fact $\quad \pi^{2k}(\mathbb{C}P^m)/\mathrm{Tor} \cong \mathbb{Z} \quad (k \leq m)$

Key Lemma Under the assumptions of Main Theorem,

$$i_1^* \otimes 1\colon \left(\pi^{2k}\left(\mathbb{C}P^{k+r(p-1)}\right) \otimes \mathbb{Z}_{(p)}\right)/\mathrm{Tor} \to \left(\pi^{2k}\left(\mathbb{C}P^{k}\right) \otimes \mathbb{Z}_{(p)}\right)/\mathrm{Tor}$$

is identified with $p^r\colon \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$

Key Lemma $\Rightarrow p^r \mid SW_X(\mathfrak{s})$.

$$
\begin{array}{ccc}
\widetilde{SW}_X(\mathfrak{s}) \in \pi^{2k}(\mathbb{C}P^{k+d/2}) & \xrightarrow{\text{hur}} & H^{2k}(\mathbb{C}P^{k+d/2}) \cong \mathbb{Z} \ni SW_X(\mathfrak{s}) \\
\downarrow i_2^* & & \cong \downarrow i_2^* \\
\pi^{2k}(\mathbb{C}P^{k+r(p-1)}) & \xrightarrow{\text{hur}} & H^{2k}(\mathbb{C}P^{k+r(p-1)}) \cong \mathbb{Z} \\
i_1^* \downarrow \times p^r & & \cong \downarrow i_1^* \\
\pi^{2k}(\mathbb{C}P^{k}) & \xrightarrow[\cong]{\text{hur}} & H^{2k}(\mathbb{C}P^{k}) \cong \mathbb{Z}
\end{array}
$$

Proof of Key lemma ($p = 2$)

The cell structure of $\mathbb{C}P^{k+1}$ is well-understood.

$k \not\equiv 0 \mod 2 \;\Rightarrow \mathbb{C}P^{k+1}/\mathbb{C}P^{k-1} \simeq S^{2k} \cup_\eta e^{2k+2}$, where η the generator of $\pi_{2k+1}(S^{2k}) \cong \mathbb{Z}/2$

Question $m\colon S^{2k} \to S^{2k}$ given. $\leftarrow$ Classified by $\pi_{2k}(S^{2k}) = \mathbb{Z}$

When does m extends to a map $\tilde{m}\colon S^{2k} \cup_\eta e^{2k+2} \to S^{2k}$?

Ans. $m \equiv 0 \mod 2$

($\because$) $m \circ \eta\colon \partial e^{2k+2} = S^{2k+1} \to S^{2k} \to S^{2k}$

$m\colon$ odd $\Rightarrow m \circ \eta \not\simeq 0 \;\Rightarrow$ cannot extend

$m\colon$ even $\Rightarrow m \circ \eta \simeq 0 \Rightarrow$ can extend

Proof of Key lemma ($p = 2$)

The cell structure of $\mathbb{C}P^{k+1}$ is well-understood.

$k \not\equiv 0 \mod 2 \Rightarrow \mathbb{C}P^{k+1}/\mathbb{C}P^{k-1} \simeq S^{2k} \cup_\eta e^{2k+2}$, where η the generator of $\pi_{2k+1}(S^{2k}) \cong \mathbb{Z}/2$

Question $m\colon S^{2k} \to S^{2k}$ given. ← Classified by $\pi_{2k}(S^{2k}) = \mathbb{Z}$

When does m extends to a map $\tilde{m}\colon S^{2k} \cup_\eta e^{2k+2} \to S^{2k}$?

Ans. $m \equiv 0 \mod 2$ →

$$\pi^{2k}\left(\mathbb{C}P^{k+1}/\mathbb{C}P^{k-1}\right)/\mathrm{Tor} \overset{\times 2}{\to} \pi^{2k}\left(\mathbb{C}P^{k}/\mathbb{C}P^{k-1}\right)/\mathrm{Tor}$$

$$\Rightarrow \pi^{2k}\left(\mathbb{C}P^{k+1}\right)/\mathrm{Tor} \overset{\times 2}{\to} \pi^{2k}\left(\mathbb{C}P^{k}\right)/\mathrm{Tor}$$

$(\because)$ $m \circ \eta\colon \partial e^{2k+2} = S^{2k+1} \to S^{2k} \to S^{2k}$

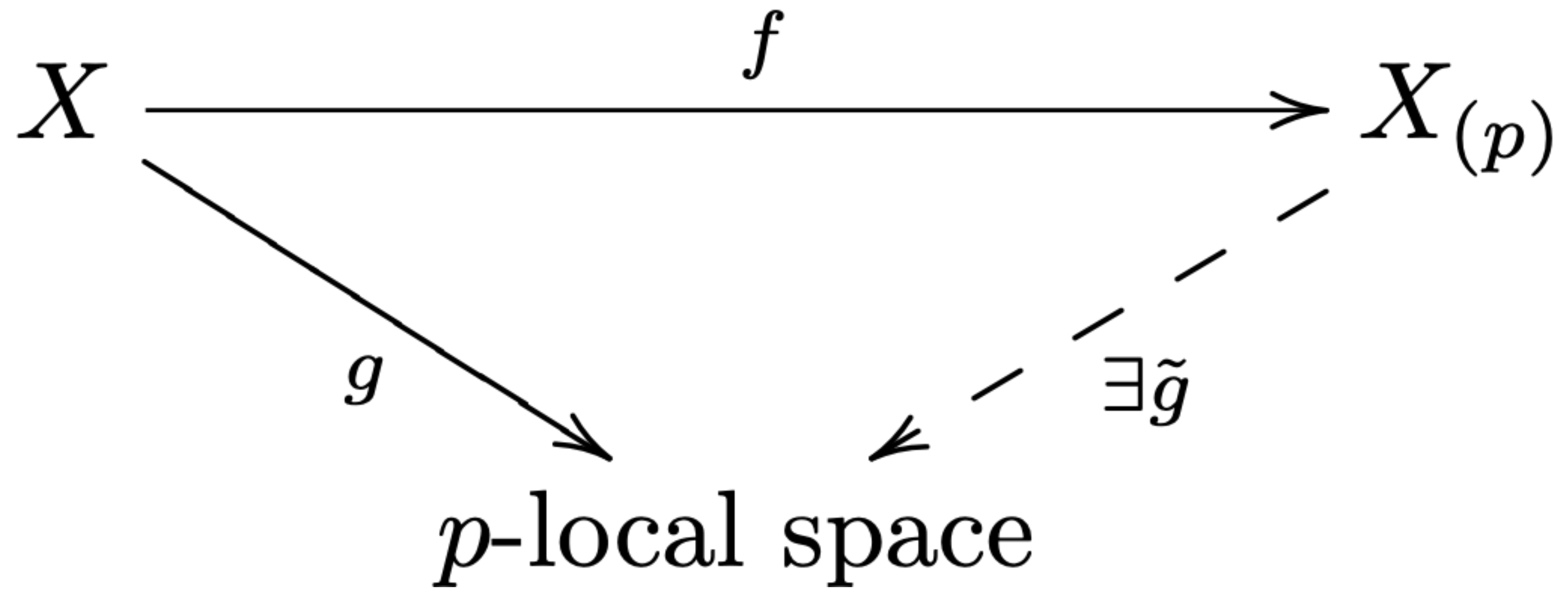
m: odd $\Rightarrow m \circ \eta \not\simeq 0 \Rightarrow$ cannot extend

m: even $\Rightarrow m \circ \eta \simeq 0 \Rightarrow$ can extend

p-localization

- Y is p-local if $\pi_*(Y)$ is a $\mathbb{Z}_{(p)}$-module.

- $f: X \to X_{(p)}$ is a p-localization if f is universal map of X into p-local spaces.



- $\pi_* \left(X_{(p)} \right) = \pi_*(X) \otimes \mathbb{Z}_{(p)}$

Theorem If X is nilpotent, then X admits a p-localization.

Corollary $\pi_1(X) = 1 \;\Rightarrow\; X$ admits a p-localization.

Telescope construction

Theorem 1 (Mimura-Nishida-Toda, 1971) There is a homotopy equivalence

$$\mathbb{C}P^n_{(p)} \simeq X^r_1 \vee \cdots \vee X^r_{p-1}$$

$$\text{s.t.} \quad X^r_i = S^{2i} \cup e^{2i+2(p-1)} \cup \cdots \cup e^{2i+2r(p-1)} \quad \text{where } r = \left[\frac{n-i}{p-1}\right]$$

Theorem 2 (Toda, 1959) For $i < p(p-1)$, there are generators α_i & α'_i s.t.

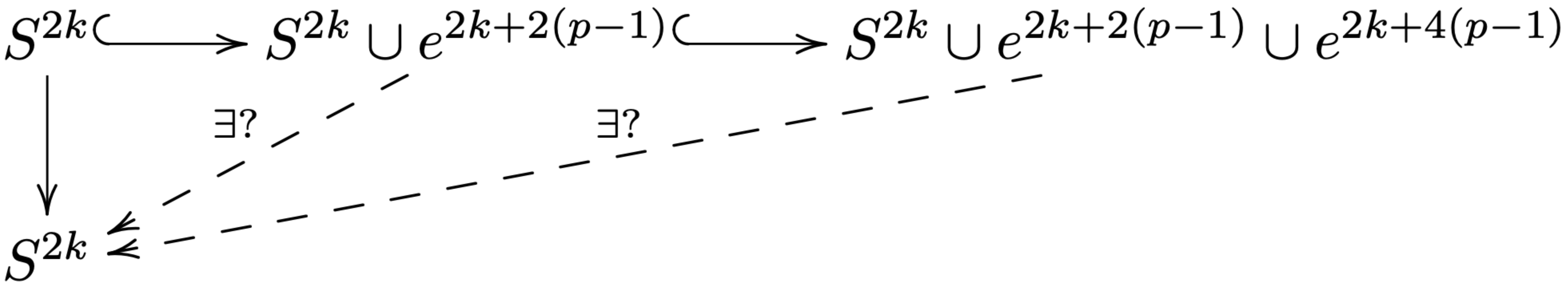
$$\pi_{2i(p-1)-1}\left(S^0\right) = \begin{cases} \mathbb{Z}/p\{\alpha_i\} & i \not\equiv 0 \mod p \\ \mathbb{Z}/p^2\{\alpha'_i\} & i \equiv 0 \mod p \end{cases}$$

Theorem 3 Let $k = i + s(p-1)$.

- $k \equiv i - s \not\equiv 0 \mod p \;\Rightarrow\; X^{s+1}/X^{s-1} \simeq S^{2k} \cup_{\alpha_1} e^{2k+2(p-1)}$
- **(D.M.Davis,1979)** $k \equiv 1 \mod p, k = ap + 1$

$$\Rightarrow X^{s+3}/X^{s-1} \simeq S^{2k} \cup_{-\alpha_1} e^{2k+2(p-1)} \cup_{\left(\frac{a}{2}+1\right)\alpha_2} e^{2k+4(p-1)} \cup_{-\frac{a+1}{2}\alpha_2+\alpha_1} e^{2k+6(p-1)}$$

- With the above ingredients, we consider the extension problem for $X^{s+t}/X^{s-1} = S^{2k} \cup e^{2k+2(p-1)} \cup e^{2k+4(p-1)} \cup \cdots \cup e^{2k+t(p-1)}$

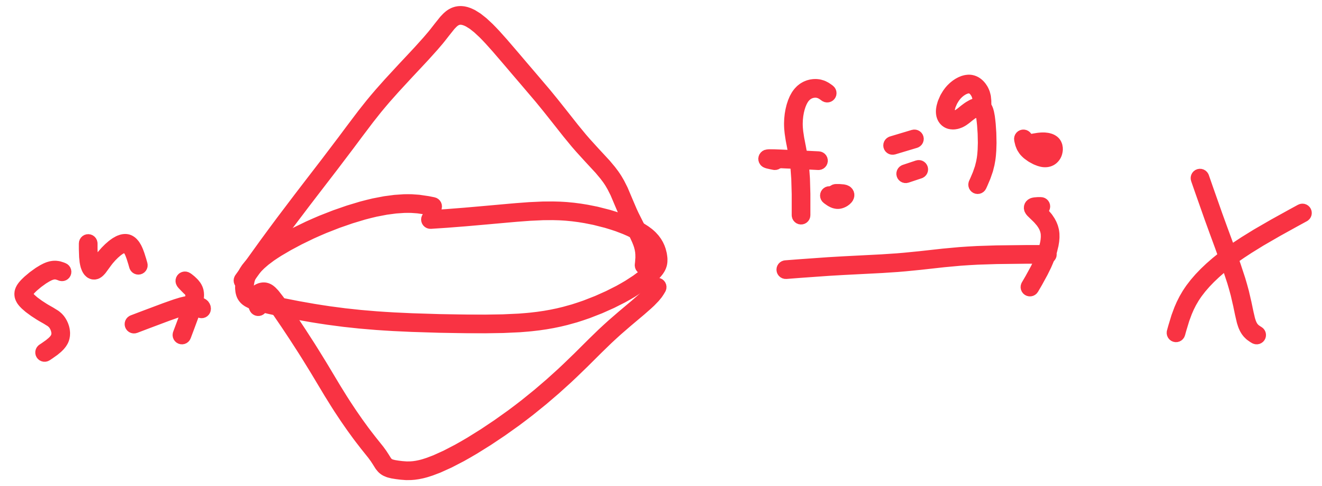


- The attaching maps are given by combinations of $\alpha_i \in \pi_{2i(p-1)-1}\left(S^0\right)$.

- We need to know the relations among α_i's → Use Toda bracket.

Toda bracket

- For two null-homotopies $f_t, g_t \colon S^n \to X$ s.t. $f_0 = g_0$ & $f_1 = g_1 = 0$, the difference obstruction $\delta(f_t, g_t) \in \pi_{n+1}(X)$ is defined



- Suppose the maps $c \colon S^q \to S^r$, $b \colon S^r \to Y$, $a \colon Y \to X$ satisfy $a \circ b \simeq 0$ & $b \circ c \simeq 0$

- Choose null-homotopies $A_t \colon S^r \to X$ & $B_t \colon S^q \to Y$ of $a \circ b$ & $b \circ c$

$\Rightarrow$ we have $\delta(a \circ B_t, A_t \circ c) \in \pi_{q+1}(X)$

- <u>Toda bracket</u>

$$\langle a, b, c \rangle = \{\delta(a \circ B_t, A_t \circ c) \text{ for all choice of } A_t,\ B_t\}$$

- Recall [Toda]. For $i < p(p-1)$, $\pi_{2i(p-1)-1}\left(S^0\right) = \begin{cases} \mathbb{Z}/p\{\alpha_i\} & i \not\equiv 0 \mod p \\ \mathbb{Z}/p^2\{\alpha_i'\} & i \equiv 0 \mod p \end{cases}$

- The generators are obtained inductively.
  - Choose a generator $\alpha_1 \in \pi_{2p-3}(S^0) = \mathbb{Z}/p$.
  - $\langle \alpha_{i-1}, p, \alpha_1 \rangle = \{\exists ! \alpha_i\}$ with $p\alpha_i = 0$
  - If $i \equiv 0 \mod p$, then $\alpha_i' = \alpha_i/p$ is a generator of $\pi_{2i(p-1)-1}\left(S^0\right) = \mathbb{Z}/p^2$.

Proposition(Toda,1959) If $s + t < p(p-1)$

$$\langle p, \alpha_s, \alpha_t \rangle = \begin{cases} \dfrac{t}{s+t}\alpha_{s+t} & s+t \not\equiv 0 \mod p \\ \dfrac{pt}{s+t}\alpha_{s+t}' & s+t \equiv 0 \mod p \end{cases}$$

Extension over two cells

- Consider a space with two cells attached: $Y \cup_{\varphi_1} e^m \cup_{\varphi_2} e^n$
- Suppose $\exists \alpha \colon Y \to S^r$ satisfying $k\alpha \circ \varphi_1 \simeq 0.$ $\qquad S^{m-1} \xrightarrow{\varphi_1} Y \xrightarrow{\alpha} S^r \xrightarrow{k} S^r$
- Then $k\alpha$ extends to a map $\widetilde{k\alpha} : Y \cup_{\varphi_1} e^m \to S^r$
- $S^{n-1} \xrightarrow{\varphi_2} Y \cup_{\varphi_1} e^m \xrightarrow{\widetilde{k\alpha}} S^r$
- Does $\widetilde{k\alpha}$ extend to a map $Y \cup_{\varphi_1} e^m \cup_{\varphi_2} e^n \to S^r$? Want to know $\widetilde{k\alpha} \circ \varphi_2 = ?.$

Lemma $\widetilde{k\alpha} \circ \varphi_2 \in \langle k, \alpha \circ \varphi_1, \Sigma^{-1}\left(\rho \circ \varphi_2\right) \rangle$

where $\rho \colon Y \cup_{\varphi_1} e^m \to (Y \cup_{\varphi_1} e^m)/Y = S^m$ is the collapsing map.

Outline of the proof of Key lemma ($p > 2$)

- Consider $X_i^{s+2}/X_i^{s-1} = S^{2k} \cup_{\varphi_1} e^{2k+2(p-1)} \cup_{\varphi_2} e^{2k+4(p-1)}$
- By Theorem 3, $k \not\equiv 0 \mod p \Rightarrow \varphi_1 = \alpha_1 \in \pi_{2p-3}(S^0) = \mathbb{Z}/p. \quad \therefore p\alpha_1 = 0.$
- Then $\theta = p \colon S^{2k} \to S^{2k}$ extends to $\tilde{\theta} \colon S^{2k} \cup_{\varphi_1} e^{2k+2(p-1)} \to S^{2k}$.
- By Lemma & [Toda], $\tilde{\theta} \circ \varphi_2 \in \langle p, \alpha_1, \alpha_1 \rangle = \alpha_2/2 \in \pi_{4(p-1)-1}(S^0) = \mathbb{Z}/p. \quad \therefore p\tilde{\theta} \circ \varphi_2 \simeq 0$
- Then $p\tilde{\theta}$ extends to $S^{2k} \cup_{\varphi_1} e^{2k+2(p-1)} \cup_{\varphi_2} e^{2k+4(p-1)} \to S^{2k}$
- The above argument implies $\left(\pi^{2k}\left(X_i^{s+2}/X^{s-1}{}_i\right) \otimes \mathbb{Z}_{(p)}\right)/\mathrm{Tor} \to \left(\pi^{2k}\left(X_i^{s}/X_i^{s-1}\right) \otimes \mathbb{Z}_{(p)}\right)/\mathrm{Tor}$ is identified with $p^2 \colon \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$.

Induction $\Rightarrow$ Key lemma

Final remarks

- Application: The adjunction inequality for negative self-intersection.
  - [Ozsvath-Szabo] supposed the simple type.
  - If $p^r \nmid SW_X(\mathfrak{s})$, we can replace the simple type assumption with a topological one.
- Future research: p-localization of the Seiberg-Witten Floer homotopy type
  - New invariant and constraint for 3-manifolds and 4-manifolds with boundary may be constructed.
  - Difficulty: computation. How?