Recent development of Seiberg-Witten Floer Theory

Homology cobordism invariants for $\mathbb{Z}HS^3$

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Recent developments of SWF theory

Homology cobordism invariants for $\mathbb{Z}HS^3$

- Froyshov invariant
- Manolescu's α, β, γ
- K-theoretic invariants

SWF stable homotopy type for Y^3 (with $b_1 > 0$)

- [Manolescu,'03] for $b_1 = 0$
- [Kronheimer-Manolescu, '02,'03,'14] for $b_1 = 1, 2$
- [T.Khandhawit-J.Lin-Sasahira,'16] for $b_1 > 0$
- [Furuta-T.Khandhawit-Matsuo-Sasahira] for $b_1 > 0$

Homology cobordism invariants for $\mathbb{Z}HS^3$ from SWF

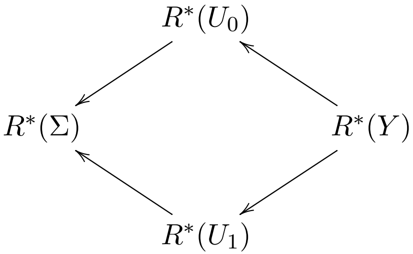
- Froyshov invariant [Froyshov'96,'10][Kronheimer-Mrowka'07]
  - Defined on Spin^c-str. $(\mathrm{U}(1))$
  - $=$ correction term of Heegaard-Floer theory
  - Definite intersection form
- Manolescu's α, β, γ [Manolescu'15]
  - Defined on Spin str. $(\mathrm{Pin}(2))$
  - Integral lifts of Rokhlin invariant.
  - β is used to disprove the Triangulation conjecture.
- K-theoretic invariants
  - Defined on Spin str. $(\mathrm{Pin}(2))$
  - [Manolescu'14] complex K_G
  - [Furuta-T.J.Li'13] complex K_G with local coefficient
  - [J.Lin'15] KO_G
  - $\frac{10}{8}$-type inequality for spin 4-manifolds with boundary

Contents

Overview Casson invariant and gauge theory

Casson invariant $\lambda(Y)$

- Y: $\mathbb{Z}HS^3$, Heegaard splitting $Y = U_0 \cup_\Sigma U_1$
- $R(X) = \{\pi_1 X \to \mathrm{SU}(2)\}/\text{conj} \cong \{\mathrm{SU}(2)\text{-flat connections}\}/\mathcal{G}$
  $R^*(X)$: irreducble part



Roughly $\boxed{\lambda(Y) = \frac{1}{2}\#(R^*(U_0) \cap R^*(U_1))}$

Chern-Simons functional & Instanton homology

- $P \to Y$: $\mathrm{SU}(2)$-bundle $\leftarrow$ Fix trivialization
- Chern-Simons functional $CS\colon \Omega^1(\mathfrak{g}_P) \to \mathbb{R}$
  where $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{su}(2)$

$$CS(A) = \frac{1}{8\pi^2} \int_Y \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

  - CS is $\mathcal{G} = \mathrm{Aut}(P)$-equivariant
  - Critical points of CS = flat connections on Y
  - grad flow $\dot{x} = -\nabla CS(x) \Leftrightarrow$ ASD eqn on $Y \times \mathbb{R}$
- [Floer'88] defined $HF^{inst}(Y) \leftarrow$ "$\infty/2$-dim" Morse homology
- [Taubes'90] $\boxed{\lambda(Y) = \frac{1}{2}\chi(HF^{inst}(Y))}$

Atiyah-Floer conjecture

Lagrangian intersection Floer homology

- (M, ω): symplectic
- $L_0, L_1 \subset M$: Lagrangian submfds
- [Floer'88]...[Fukaya-Oh-Ohta-Ono'10] $HF(L_0, L_1)$

- $R(\Sigma)$ has a symplectic structure outside singularity
- $R(U_0), R(U_1)$: Lagrangian in $R(Y)$.

Atiyah-Floer conjecture $HF^{inst}(Y) \cong HF(R(U_0), R(U_1))$

[Fukaya'15] $\mathrm{SO}(3)$-version of the Atiyah-Floer conjecture is true.

Chern-Simons-Dirac functional

- Y: closed oriented Riemannian 3-mfd (with $b_1 = 0$)
- $\mathfrak{s}$: Spin^c-str on Y, a reference Spin^c connection B_0 fixed
- Chern-Simons-Dirac functional $CSD\colon i\Omega^1(Y) \oplus \Gamma(S) \to \mathbb{R}$

$$CSD(a, \phi) = \frac{1}{2}\left(-\int_Y a \wedge da + \int_Y \langle \phi, D_a \phi \rangle \mathrm{dvol}\right)$$

where ϕ: spinor, D_a: Dirac operator

  - CSD is $\mathcal{G} = \mathrm{Map}(Y, \mathrm{U}(1))$-equivariant
  - $\nabla CSD = 0 \Leftrightarrow$ SW eqn on $(Y, \mathfrak{s})$
  - grad flow $\dot{x} = -\nabla CSD(x) \Leftrightarrow$ SW eqn on $Y \times \mathbb{R}$

Monopole(Seiberg-Witten) Floer homology

Three flavors of Monopole Floer homology [Kronheimer-Mrowka'07]

$$\widecheck{HM}(Y,\mathfrak{s}),\quad \widehat{HM}(Y,\mathfrak{s}),\quad \overline{HM}(Y,\mathfrak{s})$$

- (Analogue of) $\mathrm{U}(1)$-equivariant homologies: Borel, coBorel, Tate
  $\to \mathbb{Z}[U]$-modules($\deg U = 2$), infinitely generated as $\mathbb{Z}$-modules

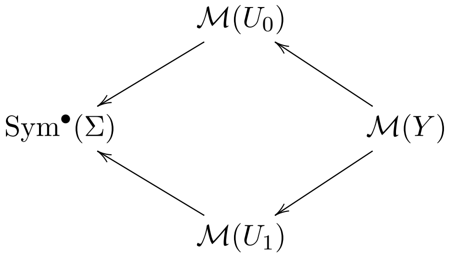
  $$\cdots \to \overline{HM}(Y,\mathfrak{s}) \to \widecheck{HM}(Y,\mathfrak{s}) \to \widehat{HM}(Y,\mathfrak{s}) \to \overline{HM}(Y,\mathfrak{s}) \to \cdots$$

- For Y: $\mathbb{Z}HS^3$, Froyshov invariant $h(Y) \in \mathbb{Z}$

$$\boxed{\lambda(Y) = \chi\left(\widecheck{HM}(Y)/\overline{HM}(Y)\right) - \frac{1}{2}h(Y)}$$

Atiyah-Floer conjecture for SWF?

- 2-dim SW eqn = vortex equation
  $\to$ moduli space $\mathcal{M}(\Sigma) = \mathrm{Sym}^\bullet(\Sigma)$.



- (Not studied yet...???)

Heegaard Floer homology

- (Σ, α, β): Heegaard diagram of 3-manifold Y

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$$

$$\mathbb{T}_\alpha \rightarrow \mathrm{Sym}^g(\Sigma) \leftarrow \mathbb{T}_\beta$$

$$\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$$

- [Ozavath-Szabo'00s] defined variants of Lagrangian Floer homology

$$HF^+(Y, \mathfrak{s}), \quad HF^-(Y, \mathfrak{s}), \quad HF^\infty(Y, \mathfrak{s})$$

$$\cdots \to HF^+(Y,\mathfrak{s}) \to HF^-(Y,\mathfrak{s}) \to HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s}) \to \cdots$$

- For Y: $\mathbb{Z}HS^3$, correction term $d(Y) \in \mathbb{Z}$
- [Ozsvath-Szabo'03]

$$\boxed{\lambda(Y) = \chi\left(HF^+(Y)/HF^\infty(Y)\right) - \frac{1}{2}d(Y)}$$

- [Kutluhan-Lee-Taubes][Colin-Ghiggini-Honda]

$$\begin{aligned} \widecheck{HM}(Y,\mathfrak{s}) &\cong HF^+(Y,\mathfrak{s}) \\ \widehat{HM}(Y,\mathfrak{s}) &\cong HF^-(Y,\mathfrak{s}) \\ \overline{HM}(Y,\mathfrak{s}) &\cong HF^\infty(Y,\mathfrak{s}) \end{aligned}$$

Seiberg-Witten-Floer homotopy type

Problem

Construct a $\mathrm{U}(1)$-space $\mathrm{SWF}(Y)$ s.t.

$$H_*^{\mathrm{U}(1)}(\mathrm{SWF}(Y)) \cong \widetilde{HM}(Y), \cdots$$

- *Cf.* [Cohen-Jones-Segal'95]
- [Manolescu'03] $b_1 = 0 \Rightarrow$ can define $\mathrm{SWF}(Y)$
- [Lidman-Manolescu'16]

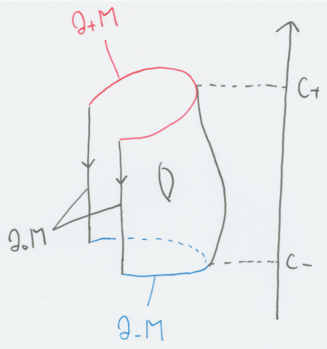
$$\begin{aligned} \widetilde{HM}(Y) &\cong H_*^{\mathrm{U}(1)}(\mathrm{SWF}(Y)) \quad \text{(Borel)} \\ \widehat{HM}(Y) &\cong cH_*^{\mathrm{U}(1)}(\mathrm{SWF}(Y)) \quad \text{(co-Borel)} \\ \overline{HM}(Y) &\cong tH_*^{\mathrm{U}(1)}(\mathrm{SWF}(Y)) \quad \text{(Tate)} \end{aligned}$$

Idea of the construction of $\mathrm{SWF}(Y)$

- Finite dimensional approximation of CSD
- Conley index

Morse theory to Conley index

- Morse function $f\colon M \to \mathbb{R}$ with $\partial M = \partial_+ M \cup \partial_0 M \cup \partial_- M$ s.t. $\partial_\pm M = f^{-1}(C_\pm)$ for $C_- < C_+$, $\partial_0 M$: along flow
$\Rightarrow \left\{ \begin{array}{l} \partial_- M \text{ : exit set} \\ \partial_+ M \text{ : entrance set} \end{array} \right\}$ of downward grad. flow



- $C_*(f)$: Morse complex

$\Rightarrow$ $H_*(C_*(f)) \cong H_*(M, \partial_- M) \cong \tilde{H}_*(M/\partial_- M)$.

- Poincaré duality
$H_*(C_*(f)) \cong H_*(M, \partial_- M) \cong H^*(M, \partial_+ M) = H^*(C^*(-f))$.

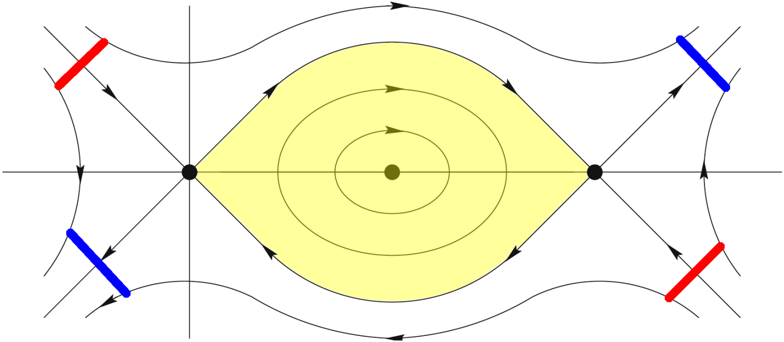
Conley index $\to$ "$M/\partial_- M$" for a dynamical system

- V: finite dim. vector space
- C^∞ complete vector field given $\to$ flow φ
- For $x \in V$, let $\mathbb{R}x$ be the oribit of x.
- $A \subset V$: compact invariant set, i.e. $\forall x \in A$, $\mathbb{R}x \subset A$.
- $U \subset V$: open nbd of A with $\overline{U}$: compact
  s.t. $\mathbb{R}x \subset U \Rightarrow \mathbb{R}x \subset A$

Theorem[Conley,Salamon,...] $U \supset \exists M \supset A$ s.t.

1. $\partial M = \underset{\text{entrance}}{\partial_+ M} \cup \underset{\text{along flow}}{\partial_0 M} \cup \underset{\text{exit}}{\partial_- M}$
2. The homotopy type of $M/\partial_- M$ $(M/\partial_+ M)$
   - depends only on A
   - is invariant under deformation of the flow φ

Conley index of (φ, A) = the pointed homotopy type of the pointed space $(M/\partial_- M, [\partial_- M])$



A: (isolated) invariant set

$\partial M = \partial_+ M \cup \partial_0 M \cup \partial_- M$

entrance along flow exit

Finite dimensional approximation of CSD

- $CSD\colon i\Omega^1(Y) \oplus \Gamma(S) \to \mathbb{R}$, $\mathcal{G}$-equivariant
- $\mathcal{G} = \mathrm{Map}(Y, \mathrm{U}(1)) = \mathcal{G}_0 \times H^1(Y;\mathbb{Z}) \times \mathrm{U}(1)$ where

$$\mathcal{G}_0 = \left\{ e^{if} \;\middle|\; f\colon Y \to \mathbb{R}, \int_Y f d\mathrm{vol} = 0 \right\} \leftarrow \text{contractible}$$

- $\mathcal{G}_0 \times H^1(Y;\mathbb{Z})$-action on $i\Omega^1(Y) \oplus \Gamma(S)$ is free.
- the slice of $\mathcal{G}_0$-action: $\mathcal{V} = i \ker d^* \oplus \Gamma(S)$
- Suppose $b_1(Y) = 0 \Rightarrow \mathcal{G} = \mathcal{G}_0 \times \mathrm{U}(1)$, CSD: $\mathcal{G}$-invariant

CSD with $\mathcal{G}$-action $\leftrightarrow$ $CSD|_{\mathcal{V}}$ with $\mathrm{U}(1)$-action

- $\dot{x} = -\nabla(CSD|_{\mathcal{V}})(x) \leftarrow \mathrm{U}(1)$-equivariant
- [Fact] $\tilde{A} = \bigcup$(bounded trajectries) $\leftarrow$ invariant set

$$b_1(Y) = 0 \quad \Rightarrow \quad \exists \text{ball} \supset \tilde{A}$$

- $\nabla(CSD|_{\mathcal{V}}) = \ell + c$,
  where $\begin{cases} \ell(a,\phi) = (*da, D_0\phi) : \text{ linear, self-adjoint} \\ c : \text{ quadratic} + \alpha \end{cases}$
- $-\tau, \nu \gg 0$, $V_\tau^\nu = \operatorname{Span}\begin{pmatrix} \text{eigenspaces of } \ell \\ \tau \leq \text{eigenvalues} < \nu \end{pmatrix}$
- Finite dimensional approximation

$$\dot{x} = -(\ell + p_\tau^\nu c)(x)$$

where $p_\tau^\nu \colon i \ker d^* \oplus \Gamma(S) \to V_\tau^\nu$, some projection

$\mathrm{SWF}(Y)$

- $A = \bigcup$(bounded trajectries of $\dot{x} = -(\ell + p^\nu_\tau c)(x)$)

$$V^\nu_\tau \supset \exists \text{ball} \supset \exists M \supset A$$

Conley index: $(M/\partial_- M, [\partial_- M])$ ← essentially $\mathrm{SWF}(Y)$

Remark

- $\mathrm{U}(1)$-action → Can take M to be a $\mathrm{U}(1)$-space
  → $\mathrm{U}(1)$-equivariant Conley index
- $(M/\partial_- M, [\partial_- M])$ depends on the choice of metric, τ, ν.
  → Introduce some $\mathrm{U}(1)$-equiv. suspension category $\mathcal{C}$
  → $\mathrm{SWF}(Y)$ is defined as an isomorphism class of objects in $\mathcal{C}$

Proposition

Let $X = M/\partial_- M$: Conley index for $\mathrm{SWF}(Y)$.

- $X^{\mathrm{U}(1)} \cong (\mathbb{R}^s)^+$ for some s.
- $\mathrm{U}(1)$ acts on $X \setminus X^{\mathrm{U}(1)}$ freely.

$(\because)$

- $\mathrm{U}(1)$-action on $\mathcal{V} = i \ker d^* \oplus \Gamma(S)$, $\begin{cases} \text{multiplication on } \Gamma(S) \\ \text{trivial on } i \ker d^* \end{cases}$
- $\mathcal{V}^{\mathrm{U}(1)} = i \ker d^* \oplus \{0\}$
- $(\ell + c)^{\mathrm{U}(1)} = \ell|_{i \ker d^* \oplus \{0\}}$
  $(\because c(a, \phi) =$ "$((\phi \otimes \phi^*)_0, a\phi)$" $= 0$ if $\phi = 0)$
- Restriction of $\dot{x} = -(\ell + c)(x)$ to $(V_\tau^\nu)^{\mathrm{U}(1)} \to \dot{x} = -\ell(x)$
- Let $s = \dim(\text{negative sp. of } \ell) \Rightarrow X^{\mathrm{U}(1)} \cong (\mathbb{R}^s)^+$

Remark: $X^{\mathrm{U}(1)} \cong (\mathbb{R}^s)^+ \leftrightarrow \overline{HM}(Y) \cong \mathbb{Z}[U, U^{-1}]$

A definition of Froyshov invariant

- Y: $\mathbb{Q}HS^3$ with Spin^c-structure
- [Froyshov invariant] $\delta^{\mathrm{U}(1)}(Y) = -h(Y) \in \mathbb{Q}$
  $\delta^{\mathrm{U}(1)}(Y)$: Manolescu's convention
  $h(Y)$: Froyshov, Kronheimer-Mrowka
- $X = M/\partial_- M$: Conley ind. for $\mathrm{SWF}(Y)$, $X^{\mathrm{U}(1)} \cong (\mathbb{R}^s)^+$.
- Apply $\tilde{H}^*_{\mathrm{U}(1)}(\,\cdot\,;\mathbb{F})$ ($\mathbb{F}$: field) to the inclusion $i\colon X^{\mathrm{U}(1)} \to X$

$$i^*\colon \tilde{H}^*_{\mathrm{U}(1)}(X;\mathbb{F}) \to \tilde{H}^*_{\mathrm{U}(1)}(X^{\mathrm{U}(1)};\mathbb{F})$$

- Note $\tilde{H}^{*+s}_{\mathrm{U}(1)}(X^{\mathrm{U}(1)}) \underset{\text{Thom iso.}}{\cong} H^*_{\mathrm{U}(1)}(pt) \cong \mathbb{F}[u]$, $\deg u = 2$.
- Via this identification, $\exists d$, $\mathrm{Im}\, i^* = \langle u^d \rangle$.

$$\boxed{\delta^{\mathrm{U}(1)}(Y) = -h(Y) = d + (\text{some grading shift})}$$

Properties of Froyshov invariant

Theorem [Froyshov]

- $\delta^{\mathrm{U}(1)}(Y_1 \# Y_2) = \delta^{\mathrm{U}(1)}(Y_1) + \delta^{\mathrm{U}(1)}(Y_2)$
- $\delta^{\mathrm{U}(1)}(-Y) = -\delta^{\mathrm{U}(1)}(Y)$
- $\delta^{\mathrm{U}(1)}(Y)$ is a Spin^c-homology cobordism invariant.

Applications of Froyshov invariant

- [Elkies]
  A symmetric unimodular form Q is definite standard $\Leftrightarrow$
  $\forall$characteristic w, $0 \geq -|w^2| + \operatorname{rank} Q$. $(*)$
- [Donaldson] For closed oriented 4-mfd Z, if the intersection form Q_Z is definite $\Rightarrow$ Q_Z: standard $\Leftrightarrow$ $(*)$.
- However if Z has a boundary Y: $\mathbb{Z}HS^3$, even when Q_Z: definite, $(*)$ may not be true.
- Instead, we can estimate how false $(*)$ is by Froyshov invariant $\delta^{\mathrm{U}(1)}(Y)$.

Theorem [Froyshov]

- Z: compact Spin^c 4-manifold s.t. $\partial Z = Y_1 \cup \cdots \cup Y_k$, Y_i: $\mathbb{Q}HS^3$.

$$b_+(Z) = 0 \quad \Rightarrow \quad \sum_{i=1}^{k} \delta^{\mathrm{U}(1)}(Y_i) \geq \frac{1}{8}(c_1(L)^2 + b_2(Z))$$

where L is the determinant line bundle.

Corollary [Froyshov]

Z: compact 4-manifold, $\partial Z = Y_1 \cup \cdots \cup Y_k$, Y_i: $\mathbb{Z}HS^3$.
If $b_+(Z) = 0$, $\Rightarrow$ for $\forall$ characteristic element w of Q_Z

$$\sum_{i=1}^{k} \delta^{\mathrm{U}(1)}(Y_i) \geq \frac{1}{8}(-|w^2| + b_2(Z))$$

Proof of Froyshov's theorem

- Y_0, Y_1: $\mathbb{Q}HS^3$ with Spin^c
- X_i: Conley ind. for $\mathrm{SWF}(Y_i)$ $(i = 0, 1)$
- Z: cobordism from Y_0 to Y_1 $\quad \partial Z = (-Y_0) \cup Y_1$
- [Manolescu][T.Khandhawit] monopole map for cobordism
  $\rightarrow$ Finite dim approx

$$f \colon \Sigma^{\bullet} X_0 \to \Sigma^{\bullet} X_1 \quad \leftarrow \mathrm{U}(1)\text{-map}$$

- We have a diagram

$$\begin{array}{ccc} \Sigma^{\bullet} X_0 & \xrightarrow{\quad f \quad} & \Sigma^{\bullet} X_1 \\ \uparrow & & \uparrow \\ (\Sigma^{\bullet} X_0)^{\mathrm{U}(1)} & \xrightarrow[\text{if } \boxed{b_+(Z) = 0}]{\cong} & (\Sigma^{\bullet} X_1)^{\mathrm{U}(1)} \end{array}$$

- Apply $\tilde{H}^*_{\mathrm{U}(1)}(\,\cdot\,)$

$$
\begin{array}{ccc}
\tilde{H}^*_{\mathrm{U}(1)}(\Sigma^\bullet X_0) & \xleftarrow{f^*} & \tilde{H}^*_{\mathrm{U}(1)}(\Sigma^\bullet X_1) \\
\downarrow & & \downarrow \\
\tilde{H}^*_{\mathrm{U}(1)}((\Sigma^\bullet X_0)^{\mathrm{U}(1)}) & \xleftarrow{\cong} & \tilde{H}^*_{\mathrm{U}(1)}((\Sigma^\bullet X_1)^{\mathrm{U}(1)}) \\
\| & & \| \\
\mathbb{F}[u] & & \mathbb{F}[u] \\
\cup & & \cup \\
\langle u^{d_0} \rangle & & \langle u^{d_1} \rangle
\end{array}
$$

$$
\Rightarrow d_0 \leq d_1 \Rightarrow \boxed{\delta^{\mathrm{U}(1)}(Y_0) + \frac{1}{8}(c_1^2(L) + b_2(Z)) \leq \delta^{\mathrm{U}(1)}(Y_1)}
$$

Spin structure

- On Spin structure, Seiberg-Witten Floer theory has a $\mathrm{Pin}(2)$-symmetry $\quad \mathrm{Pin}(2) = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}$
- In fact, $G = \mathrm{Pin}(2)$-action on $\ker d^* \oplus \Gamma(S)$ is given
  $$\begin{cases} \text{on } \Gamma(S) \text{ by multiplication} \\ \text{on } \ker d^* \text{ via } \mathrm{Pin}(2) \to \{\pm 1\},\ j \mapsto -1 \end{cases}$$
  $\Rightarrow \ell + c$: G-equivariant
- We obtain $\mathrm{SWF}(Y)$ with G-action whose X satisfies
  - $X^{\mathrm{U}(1)} \cong (\tilde{\mathbb{R}}^s)^+$ where $\tilde{\mathbb{R}} = (\mathrm{Pin}(2) \to \{\pm 1\} \curvearrowright \mathbb{R})$
  - G acts freely on $X \setminus X^{\mathrm{U}(1)}$
- For a spin cobordism Z from Y_0 to Y_1, we have G-equivariant finite dim approx
  $$f\colon \Sigma^{\bullet} X_0 \to \Sigma^{\bullet} X_1$$

Manolescu's α, β, γ

- [Fact]

$$H^*_G(pt;\mathbb{F}_2) = H^*(BG;\mathbb{F}_2) = \mathbb{F}_2[q,v]/(q^3)$$

where $\deg q = 1$, $\deg v = 4$
$(\because G \to \mathrm{SU}(2) \to \mathbb{RP}^2 \to BG \to B\,\mathrm{SU}(2) \cong \mathbb{HP}^\infty)$

- Apply $\tilde{H}^*_G(\,\cdot\,;\mathbb{F}_2)$ to the inclusion $(\tilde{\mathbb{R}}^s)^+ \cong X^{\mathrm{U}(1)} \xrightarrow{i} X$

$$\tilde{H}^*_G(X) \xrightarrow{i^*} \tilde{H}^*_G(X^{\mathrm{U}(1)}) \cong \tilde{H}^*_G((\tilde{\mathbb{R}}^s)^+) \underset{\text{Thom iso.}}{\cong} H^*_G(pt;\mathbb{F}_2)$$

- $\operatorname{im} i^*$ can be identified with an ideal $\mathcal{J} \subset \mathbb{F}_2[q,v]/(q^3)$
- $\mathcal{J}$ has a form (v^a, qv^b, q^2v^c) for $a \geq b \geq c \geq 0$

Example of $\mathcal{J} = (v^a, qv^b, q^2v^c)$

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & q^2v & q^2v^2 & q^2v^3 & q^2v^4 & q^2v^5 & \cdots \\
0 & 0 & 0 & qv^3 & qv^4 & qv^5 & \cdots \\
0 & 0 & 0 & 0 & v^4 & v^5 & \cdots \\
\hline
 & \downarrow & & \downarrow & \downarrow & & \\
 & c=1 & & b=3 & a=4 & &
\end{array}
$$

- $\alpha(Y)$, $\beta(Y)$, $\gamma(Y)$ are defined as $4a$, $4b$, $4c$ with some grading shift

Properties of $\alpha(Y)$, $\beta(Y)$, $\gamma(Y)$

- $\alpha(Y)$, $\beta(Y)$, $\gamma(Y) \in \frac{1}{8}\mathbb{Z}$ for $\mathbb{Q}HS^3$ ($\in \mathbb{Z}$ for $\mathbb{Z}HS^3$)
- $\alpha(Y) \geq \beta(Y) \geq \gamma(Y)$
- $\alpha(Y) \equiv \beta(Y) \equiv \gamma(Y) \equiv \mu(Y)$ (mod 2) where $\mu(Y)$: Rokhlin
- $\alpha(-Y) = -\gamma(Y)$, $\boxed{\beta(-Y) = -\beta(Y)}$, $\gamma(-Y) = -\alpha(Y)$
- Z: oriented cobordism from Y_0 to Y_1, $b_+(Z) = 0$

$$\alpha(Y_1) \geq \alpha(Y_0) + \frac{1}{8}b_2(Z)$$
$$\beta(Y_1) \geq \beta(Y_0) + \frac{1}{8}b_2(Z)$$
$$\gamma(Y_1) \geq \gamma(Y_0) + \frac{1}{8}b_2(Z)$$

In particular, $\alpha(Y)$, $\beta(Y)$, $\gamma(Y)$ are homoogy cobordism invariants

Disproof of the Triangulation conjecture

Proposition[Manolescu]

If Y is a $\mathbb{Z}HS^3$ with $\mu(Y) = 1$, then $Y\#Y$ is not homology cobordant to S^3.

Proof.

If $Y\#Y \underset{\text{h.cob}}{\sim} S^3 \Rightarrow Y \underset{\text{h.cob}}{\sim} -Y \Rightarrow \beta(Y) = \beta(-Y) = -\beta(Y)$.
$\therefore \beta(Y) = 0$. $\therefore \mu(Y) \underset{(2)}{\equiv} \beta(Y) = 0$. $\square$

Corllary

For $\forall n \geq 5$, $\exists$closed topological n-manifold which does not admit a simplicial triangulation.

[Matumoto'76][Galewski-Stern'77] Proposition $\Rightarrow$ Corollary

Applications by Stoffregen

- [Stoffregen'15] studies α, β, γ for connected sums of $\mathbb{Q}HS^3$, especially Seifert fibered spaces

Theorem[Stoffregen'15]

The integral homology cobordism group θ^3_H contains a $\mathbb{Z}^\infty$ summand generated by

$$\Sigma(p, 2p-1, 2p+1), \quad p \geq 3, \ p: \ \text{odd}$$

Cf. [Furuta'90](p=2, q=3), [Fintushel-Stern'90]

$$\Sigma(p, q, pqn-1), \quad n \geq 1, \ p, q: \ \text{relatively prime}$$

are linearly independent in θ^3_H

K-theoretic invariant $\kappa(Y)$ [Manolescu'14]

- $h = (G = \mathrm{Pin}(2) \underset{\text{multiplication}}{\curvearrowright} \mathbb{H})$, $z = e(\mathbb{H}) = \Lambda_{-1}h = 2 - h$
- $c = (G \to \{\pm 1\} \curvearrowright \tilde{\mathbb{C}})$, $\quad w = e(\tilde{\mathbb{C}}) = \Lambda_{-1}c = 1 - c$.
- $R(G) = \mathbb{Z}[z, w]/(w^2 - 2w, zw - 2w)$
- $\mathrm{SWF}(Y)$ for spin Y. Can take X s.t. $X^{\mathrm{U}(1)} = (\tilde{\mathbb{C}}^s)^+$.

$$\tilde{K}_G(X^{\mathrm{U}(1)}) = K_G(\tilde{\mathbb{C}}^s) \underset{\text{Thom iso}}{\cong} R(G)$$

- For the inclusion $i\colon X^{\mathrm{U}(1)} \to X$

$$\tilde{K}_G(X) \xrightarrow{i^*} \tilde{K}_G(X^{\mathrm{U}(1)}) = R(G) \xrightarrow{\mathrm{tr}_j} \mathbb{Z}$$

Then the image is an ideal of the form (2^k)

$$\boxed{\kappa(Y) = k + (\text{grading shift})}$$

Properties of $\kappa(Y)$

- $\kappa(Y) \equiv \mu(Y)$ mod 2
- Z: cobordism from Y_0 to Y_1 with $b_+(Z) = 0$

$$\Rightarrow \quad \kappa(Y_1) \geq \kappa(Y_0) + \frac{1}{8} b_2(Z)$$

- Z: spin cobordism from Y_0 to Y_1 with intersection form $Q_Z = p(-E_8) \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow \quad \kappa(Y_1) + q \geq \kappa(Y_0) + p - 1$$

Cf. [Furuta] $Y_0 = Y_1 = S^3$, $q \geq 1 \Rightarrow q \geq p + 1$

Other versions of K-theoretic invariants

- [Furuta-T.J.Li'13] complex K_G with local coefficient
- [J.Lin'15] KO_G

Subgroups of $G = \mathrm{Pin}(2)$ & Involutive Heegaard Floer homology

- For a subgroup $I \subset G$, we can apply H^*_I or K_I to $\mathrm{SWF}(Y)$ $\to$ invariants analogous to δ, α, β, γ, κ.
- $I = \langle j \rangle = \mathbb{Z}/4 \Rightarrow H^*_{\mathbb{Z}/4}(\mathrm{SWF}(Y))$, $\underline{\delta}(Y)$, $\overline{\delta}(Y)$

[Hendricks-Manolescu'15] Involutive Heegaard Floer homology

$$HFI^{\circ}(Y), \quad \circ \in \{\hat{}, +, -, \infty\}$$

correction terms $\underline{d}(Y), \overline{d}(Y)$

Defined by using the symmetry $(\Sigma, \alpha, \beta) \leftrightarrow (-\Sigma, \beta, \alpha)$

Conjecture

$$HFI^{+}(Y) \cong H^{\mathbb{Z}/4}_{*}(\mathrm{SWF}(Y); \mathbb{Z}/2), \dots$$

$$\underline{d}(Y) = \underline{\delta}(Y), \quad \overline{d}(Y) = \overline{\delta}(Y)$$

$\mathbb{Z}_2$-Froyshov invariant

- Y: $\mathbb{Q}HS^3$ with Spin^c-structure
- $I = \{\pm 1\} \subset S^1$
  $\Rightarrow$ Can define $\mathbb{Z}_2$-Froyshov invariant $\delta^{\mathbb{Z}_2}(Y) \in \mathbb{Q}$
- [Stoffregen'16] $\delta^{\mathbb{Z}_2}(Y) = \delta^{\mathrm{U}(1)}(Y)$

Theorem [N.'16]

W: cobordism from Y_0 to Y_1 with a $\mathbb{Z}$ bundle $\ell \to W$ s.t.
$b_+^\ell(W) = 0$ & $\ell|_{Y_0}, \ell|_{Y_1}$: trivial where $b_\circ(W) = \dim H_\circ(W;\ell)$
For a class $C \in H^2(W;\ell)$ s.t. $C \underset{(2)}{\equiv} w_2(X) + w_1(\ell \otimes \mathbb{R})^2$,

$$\delta^{\mathbb{Z}_2}(Y_1) \geq \delta^{\mathbb{Z}_2}(Y_0) + \frac{1}{4}(C^2 + b_2^\ell(W))$$

The proof uses $\mathrm{Pin}^-(2)$-monopole equations

Corollary [N.'16]

W: compact 4-manifold, $\partial W = Y_1 \cup \cdots \cup Y_k$, Y_i: $\mathbb{Z}HS^3$.
Suppose a $\mathbb{Z}$ bundle $\ell \to W$ satisfies

- $w_1(\ell \otimes \mathbb{R})^2 = 0$
- ℓ-coefficient intersection form $Q_{W,\ell}$ is definite

Then, for $\forall$characteristic element w of $Q_{W,\ell}$

$$\sum_{i=1}^{k} \delta^{\mathrm{U}(1)}(Y_i) = \sum_{i=1}^{k} \delta^{\mathbb{Z}/2}(Y_i) \geq \frac{1}{8}(-|w^2| + b_2^{\ell}(W))$$

(*Cf.*) [N.'13] X: closed oriented 4-manifold with a local system ℓ s.t. $w_1(\ell \otimes \mathbb{R})^2 = 0$ & $Q_{X,\ell}$: definite

$$\Rightarrow \quad 0 \geq \frac{1}{8}(-|w^2| + b_2^{\ell}(W)) \text{ for } \forall w\text{: characteristic}$$

$$\underset{\text{[Elkies]}}{\Rightarrow} \quad Q_{X,\ell} \cong \text{ diagonal}$$