$\mathrm{Pin}^{-}(2)$-monopole equations and Yamabe invariants

(Joint work with M. Ishida & S. Matsuo)

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Nov 22, 2014

Introduction

- M: closed oriented connected n-manifold ($n \geq 3$) $\leftarrow$ always assumed
- g: Riemannian metric on M
- $s_g \colon M \to \mathbb{R}$: scalar curvature
- $[g] = \{ug \mid u \colon M \to \mathbb{R}^+\}$, conformal class of g
- $\mathcal{C}(M)$: the space of conformal classes

Yamabe invatiant

$$\mathcal{Y}(M) = \sup_{[g]\in\mathcal{C}(M)} \inf_{h\in[g]} \frac{\int_M s_h d\mu_h}{V_h^{\frac{n-2}{n}}},$$

where $d\mu_h$ is the volume form of h, and $V_h = \int_M d\mu_h$.

In general, it is difficult to compute the exact values of the Yamabe invariants.

Theorem([LeBrun,'96,'99])

Let M be a compact minimal Kähler surface s.t. $b_+ \geq 2$, $c_1^2(M) \geq 0$. Then

$$\mathcal{Y}(M) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$

- The proof uses the Seiberg-Witten equations.
  A key point is the nontriviality of the SW invariant.

\- Note $c_1^2(M) = 2\chi(M) + 3\tau(M)$. χ: Euler, τ: signature

Main theorem

Theorem 1 (Ishida-Matsuo-N.,'14)

Let M be a compact minimal Kähler surface s.t. $b_+ \geq 2$, $c_1^2(M) \geq 0$.
Let $Z = Z_1 \# \cdots \# Z_k$ such that

$$Z_i = \overline{\mathbb{CP}}^2 \quad \text{or} \quad S^2 \times \Sigma \quad \text{or} \quad S^1 \times N^3$$

with $g(\Sigma) > 0$ & $\mathcal{Y}(N) \geq 0$. (Ex. $N = S^3$, $L(p,q)$, $S^1 \times \Sigma, \ldots$)

Then $$\mathcal{Y}(M \# Z) = \mathcal{Y}(M) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$

- The proof uses the $\mathrm{Pin}^-(2)$-monopole equations.
  (*Cf.* SW equations $=$ $\mathrm{U}(1)$-monopole equations)
- If $\exists Z_i = S^2 \times \Sigma$ or $S^1 \times N$ s.t $b_1(N) \geq 1$ $(b_+(Z_i) \geq 1)$
  $\Rightarrow$ SW inv of $M \# Z \equiv 0$.
  But $\mathrm{Pin}^-(2)$-monopole inv $\mathrm{SW}^{\mathrm{Pin}^-(2)}(M \# Z) \not\equiv 0$.

Contents

Yamabe invariants

Reference
[K. Akutagawa] *Yamabe invariants*, Sugaku, 66-1(2014), 31–60.

- M: closed oriented connected n-manifold ($n \geq 3$)
- $\mathcal{M}(M) =$ the space of Riemannian metrics on M.

(Normalized) Einstein-Hilbert functional

$$E\colon \mathcal{M}(M) \to \mathbb{R}, \quad g \mapsto \frac{\int_M s_g d\mu_g}{V_g^{\frac{n-2}{n}}}$$

- Critical points of E are Einstein metrics.
- $\inf_g E(g) = -\infty, \quad \sup_g E(g) = +\infty.$

- $\mathcal{M}_1(M) = \{g \in \mathcal{M}(M) \mid V_g = 1\}, \quad [g]_1 = [g] \cap \mathcal{M}_1(M)$.
- For $h \in \mathrm{Crit}\left(E|_{\mathcal{M}_1(M)}\right)$, if $(M, h) \not\cong (S^n, g_0)$ (not conformal iso.), then

$$T_h\mathcal{M}_1(M) \cong T_h(\mathrm{Diff}(M)^* h) \oplus T_h[h]_1 \oplus {}^{\exists}\mathbb{V}.$$

- $\exists W \subset \mathbb{V}$: finite dimensional subspace, for smooth variation $\{h_t\}_{t\in(-\varepsilon,\varepsilon)}$,

$$\left.\frac{d^2}{dt^2}E(h_t)\right|_{t=0} \begin{cases} = 0 & \text{if } \dot{h}_t(0) \in T_h(\mathrm{Diff}(M)^* h) \\ > 0 & \text{if } \dot{h}_t(0) \in T_h[h]_1 \\ < 0 & \text{if } \dot{h}_t(0) \in \mathbb{V} \cap W^{\perp} \end{cases}$$

- Try the min-max approach to find saddle points!

Yamabe constant

- Yamabe constant is defined via the min process.
- [Fact] For any conformal class $[g]$, $E|_{[g]}$ is bounded below.

Yamabe constant $Y(M,[g]) = \inf_{h\in[g]} E(h)$

Theorem (Yamabe, Trudinger, Aubin, Schoen,...)

$\forall$compact manifold M, $\forall$conformal class $[g]$, $\exists h_0 \in [g]$ s.t.

$$E(h_0) = \inf_{h\in[g]} E(h) = Y(M,[g]).$$

- h is called the Yamabe metric. Then

$$s_h = Y(M,[g]) \cdot V_h^{-\frac{2}{n}} \longleftarrow \text{const.}$$

Yamabe invariant

- Next consider the max process.

Definition (O. Kobayashi, Schoen, around '85)

$$\mathcal{Y}(M) = \sup_{[g]\in\mathcal{C}(M)} Y(M,[g])$$

Kobayashi says: "I wanted to consider something like the curvature of a smooth manifold. The Yamabe invariant is one of such things."

- If $C \in \mathcal{C}(M)$ attains the sup, and $\mathcal{Y}(M) \leq 0$,
  $\Rightarrow$ the Yamabe metric of C is Einstein.
- But the sup is not necessarily attained in general.

Properties

- Aubin's inequality $\forall M^n$, $\mathcal{Y}(M) \leq \mathcal{Y}(S^n) = \mathcal{Y}(S^n, [h_0])$.
- $\mathcal{Y}(M)$ is a diffeomorphism invariant.
- $n = 3 \Rightarrow \mathcal{Y}(M)$ is a topological invariant.
- $n \geq 4 \Rightarrow \mathcal{Y}(M)$ depends on differential structures.

Ex.
- $\mathcal{Y}(\mathbb{C}\mathrm{P}^2 \# 9\overline{\mathbb{C}\mathrm{P}}^2) > 0$
- $\mathcal{Y}(\text{Dolgachev surface}) = 0$
- $\mathbb{C}\mathrm{P}^2 \# 9\overline{\mathbb{C}\mathrm{P}}^2 \underset{\text{homeo}}{\cong} (\text{Dolgachev surface})$.

- $\mathcal{Y}(M) > 0 \Leftrightarrow M$ admits a positive scalar curvature (PSC) metric.
- Hence, if $\mathcal{Y}(M) > 0 \Rightarrow$ the ordinary & stable cohomotopy SW and $\mathrm{Pin}^-(2)$-monopole invariants vanish.

Yamabe invariants of 4-manifolds

- $\mathcal{Y}(S^4) = Y(S^4, [h_0]) = 8\sqrt{6}\pi$
- [LeBrun,'96,'99]
  For a compact minimal Kähler surface M s.t. $b_+ \geq 2$, $c_1^2(M) \geq 0$,

$$\mathcal{Y}(M) = \mathcal{Y}(M \# k\overline{\mathbb{C}\mathrm{P}}^2) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$

- [LeBrun,'99]
  For a compact Kähler surface M

$$\begin{cases} \mathcal{Y}(M) > 0 & \Leftrightarrow \quad \mathrm{Kod}(M) = -\infty \\ \mathcal{Y}(M) = 0 & \Leftrightarrow \quad \mathrm{Kod}(M) = 0, 1 \\ \mathcal{Y}(M) < 0 & \Leftrightarrow \quad \mathrm{Kod}(M) = 2 \end{cases}$$

- [Ishida-LeBrun,'03]
  For compact Kähler surfaces M_i $(i=1,2,3,4)$ s.t. $b_1(M_i)=0$ + some conditions,

$$\mathcal{Y}(\#_{i=1}^k M_i) = \mathcal{Y}(\#_{i=1}^k M_i \# k\overline{\mathbb{C}\mathrm{P}}^2) = -4\sqrt{2}\pi\sqrt{\sum_{i=1}^k c_1^2(M_i)}.$$

  The proof uses the stable cohomotopy SW.

- [Sasahira,'06]
  M_1, M_2, $M_3 = K3$ or $\Sigma_g \times \Sigma_h$ $(g,h$:odd$)$

$$\mathcal{Y}(\#_{i=1}^k M_i) = \mathcal{Y}(\#_{i=1}^k M_i \# k\overline{\mathbb{C}\mathrm{P}}^2) = -4\sqrt{2}\pi\sqrt{\sum_{i=1}^k c_1^2(M_i)}.$$

  The proof uses the stable cohomotopy SW.

[Sung,'09]

- M: a compact Kähler s.t. $\mathrm{Kod}(M) \geq 0$
  N: $b_+(N) = 0$ & $\mathcal{Y}(N) \geq 0$

$$\mathcal{Y}(M \# N) = \mathcal{Y}(M).$$

- M, N: as above.
  Let $C_1 \subset M$ & $C_2 \subset N$ be circles s.t. $[C_2] \neq 0$ in $H_1(N; \mathbb{R})$.
  Let $n(C_i)$ be the tubular nbd of C_i.
  Let $\tilde{M} = (M \setminus n(C_1)) \bigcup_{\partial n(C_1) = \partial n(C_2)} (N \setminus n(C_2))$.

$$\mathcal{Y}(\tilde{M}) = \mathcal{Y}(M)$$

The proof uses the (ordinary) SW.

[Gursky-LeBrun,'98]

$k = 1, 2, 3, \quad \forall l,$

$$\mathcal{Y}(\mathbb{C}\mathrm{P}^2 \# l(S^1 \times S^3)) = \mathcal{Y}(\mathbb{C}\mathrm{P}^2) = 12\sqrt{2}\pi < \mathcal{Y}(S^4) = 8\sqrt{6}\pi$$

$$0 < \mathcal{Y}(k\,\mathbb{C}\mathrm{P}^2 \# l(S^1 \times S^3)) \leq 4\pi\sqrt{2k+16} < \mathcal{Y}(S^4)$$

Proof uses

- modified scalar curvature,
- conformal scaling trick,
- $\mathrm{ind}\, D_A > 0 \Rightarrow \exists \Phi$ s.t. $D_A \Phi = 0$ & Weitzenböck formula

Outline of the proof

Our main theorem, again.

Theorem 1 (Ishida-Matsuo-N.,'14)

Let M be a compact minimal Kähler surface s.t. $b_+ \geq 2$, $c_1^2(M) \geq 0$.
Let $Z = Z_1 \# \cdots \# Z_k$ such that

$$Z_i = \overline{\mathbb{C}\mathrm{P}}^2 \quad \text{or} \quad S^2 \times \Sigma \quad \text{or} \quad S^1 \times N^3$$

with $g(\Sigma) > 0$ & $\mathcal{Y}(N) \geq 0$. (Ex. $N = S^3$, $L(p,q)$, $S^1 \times \Sigma, \ldots$)

Then $$\mathcal{Y}(M \# Z) = \mathcal{Y}(M) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$

Introduce the following invariant:

$$\mathcal{I}_s(M) = \inf_{g \in \mathcal{M}(M)} \int_M |s_g|^{\frac{n}{2}} d\mu_g$$

Theorem

- [Kobayashi,'90][Besson-Curtois-Gallot, '91]

$$\mathcal{I}_s(M) = \begin{cases} 0 & \text{if } \mathcal{Y}(M) \geq 0 \\ |\mathcal{Y}(M)|^{\frac{n}{2}} & \text{if } \mathcal{Y}(M) \leq 0 \end{cases}$$

- [Kobayashi,'87]

$$\mathcal{I}_s(M \# N) \leq \mathcal{I}_s(M) + \mathcal{I}_s(N)$$

For our $M\#Z = M\#Z_1\#\cdots\#Z_k$, if $\mathcal{Y}(M\#Z) \leq 0$, then

$$
\begin{aligned}
|\mathcal{Y}(M\#Z)|^2 =& \mathcal{I}_s(M\#Z_1\#\cdots\#Z_k) \\
\leq& \mathcal{I}_s(M) + \mathcal{I}_s(Z_1) + \cdots + \mathcal{I}_s(Z_k).
\end{aligned}
$$

- $\mathcal{I}_s(M) = 32\pi c_1^2(M)$ by [LeBrun]
- $Z_i = \overline{\mathbb{C}\mathrm{P}}^2$ or $S^2 \times \Sigma \Rightarrow \mathcal{I}_s(Z_i) = 0$
  ($\because$ Z_i admits a PSC metric $\Leftrightarrow \mathcal{Y}(Z_i) > 0 \Rightarrow \mathcal{I}_s(Z_i) = 0$.)
- $\mathcal{Y}(N^3) \geq 0 \underset{\text{[Petean]}}{\Rightarrow} \mathcal{Y}(S^1 \times N) \geq 0 \Rightarrow \mathcal{I}_s(S^1 \times N) = 0$.

$$
\boxed{\mathcal{Y}(M\#Z) \leq 0 \Rightarrow |\mathcal{Y}(M\#Z)|^2 = \mathcal{I}_s(M\#Z) \leq 32\pi c_1^2(M)}
$$

Now, to prove is

$$
\mathcal{Y}(M\#Z) \leq 0 \quad \& \quad |\mathcal{Y}(M\#Z)|^2 = \mathcal{I}_s(M\#Z) \geq 32\pi c_1^2(M)
$$

These are proved by using $\mathrm{Pin}^-(2)$-monopole equations

$\mathrm{Pin}^-(2)$-monopole equations

Spin^{c-}-structure

- $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$,
  $\mathrm{Pin}^-(2) = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1)$
- $\mathrm{Spin}^{c-}(4)/\mathrm{Pin}^-(2) = \mathrm{Spin}(4)/\{\pm 1\} = \mathrm{SO}(4)$
- $\mathrm{Spin}^{c-}(4) \supset \mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$
  $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}^c(4) = \{\pm 1\}$.

Let X be a closed ori. Riemannian 4-manifold with (nontrivial) double covering $\tilde{X} \overset{2:1}{\to} X$

Definition

Spin^{c-}-structure on $\tilde{X} \to X$ is a $\mathrm{Spin}^{c-}(4)$-bundle P over X with

$$P/\mathrm{Pin}^-(2) \overset{\cong}{\to} Fr(X), \quad P/\mathrm{Spin}^c(4) \overset{\cong}{\to} \tilde{X}$$

Characteristic $\mathrm{O}(2)$-bundle

- $\mathrm{Spin}^{c-}(4)/\,\mathrm{Spin}(4) = \mathrm{Pin}^-(2)/\{\pm 1\} = \mathrm{O}(2)$

$\Rightarrow E = P/\,\mathrm{Spin}(4)$ is an $\mathrm{O}(2)$-bundle
$\rightarrow$ Characteristic $\mathrm{O}(2)$-bundle

$$\tilde{c}_1(E) \in H^2(X;l), \qquad \text{where } l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}.$$

$\mathrm{Pin}^-(2)$-monopole equations

$$\begin{cases} D_A\Phi = 0, \\ F_A^+ = q(\Phi), \end{cases}$$

where A: $\mathrm{O}(2)$-connection on E, Φ: positive spinor.

- $b^l_+ := \dim H_+(X; l \otimes \mathbb{R}) \geq 2$
  $\Rightarrow \mathrm{Pin}^-(2)$-monopole invariant $\mathrm{SW}^{\mathrm{Pin}^-(2)}$ is defined.

Proposition 1 $\quad \mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0 \Rightarrow \mathcal{Y}(M) \leq 0.$
($\because \mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0 \Rightarrow$ No PSC metric $\Leftrightarrow \mathcal{Y}(M) \leq 0.$)

- Let $a \in \Omega^2(l \otimes \sqrt{-1}\mathbb{R})$ be the g-harmonic representative of $\tilde{c}_1(E)$.
- Decompose a into the g-self-dual & g-anti-self-dual parts:

$$a = a_+ + a_-$$

Basic estimate 1

$$\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0 \Rightarrow \int_X |s_g|^2 d\mu_g \geq 32\pi^2 (a_+)^2 \quad \text{for } \forall g \in \mathcal{M}(X)$$

$(\because)$ The Weitzenböck formula.

$$|D_A\Phi|^2 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s_g|\Phi|^2 + \langle F_A^+ \cdot \Phi, \Phi\rangle.$$

$$(A, \Phi) : \text{a solution } \Rightarrow 0 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s_g|\Phi|^2 + |\Phi|^4$$

$$\int |\Phi|^4 d\mu \leq \int (-s_g)|\Phi|^2 d\mu \leq \left(\int |s_g|^2 d\mu\right)^{\frac{1}{2}} \left(\int |\Phi|^4 d\mu\right)^{\frac{1}{2}}$$

$$\int |s_g|^2 d\mu \geq \int |\Phi|^4 d\mu = 8\int |F_A^+|^2 d\mu = 32\pi^2 (a_+)^2.$$

Corollary $\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0 \Rightarrow \mathcal{I}_s(X) \geq 32\pi^2 a_+^2$

Non-vanishing theorem

$\exists\, \mathrm{Spin}^{c-}$-structure on $M\#Z$ s.t.

- $\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0$
- $\forall g \quad (a_+)^2 \geq c_1^2(M)$
  where a_+ is the g-self-dual part of the g-harmonic form a representing $\tilde{c}_1(E)$.

From these, we obtain

$$\mathcal{Y}(M\#Z) \leq 0 \quad \& \quad |\mathcal{Y}(M\#Z)|^2 = \mathcal{I}_s(M\#Z) \geq 32\pi c_1^2(M)$$

Theorem 1 is proved.

Obstructions to the existence of Einstein metrics & Ricci flow

- A Riemannian metric g is Einstein $\Leftrightarrow \mathrm{Ric}_g = \lambda g$, $\lambda \in \mathbb{R}$.
- $\{g_t\}$: 1-parameter family of Riemannian metrics
  Normalized Ricci flow

$$\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}_g + \frac{2}{n}\left(\frac{\int_M s_g d\mu_g}{V_g}\right) g$$

- When h is Einstein, the constant family $g_t = h$ is a solution of the normalized Ricci flow after rescaling.

Let $M\#Z = M\#Z_1\#\cdots\#Z_k$ be as in Theorem 1.

Theorem 2 (Ishida-Matsuo-N)

If $M\#Z$ admits an Einstein metric

$$\Rightarrow \qquad 4-(2\chi(Z)+3\tau(Z)) < \frac{1}{3}(2\chi(M)+3\tau(M)).$$

Cf. Hitchin-Thorpe inequality Einstein $\Rightarrow 2\chi(X) - 3|\tau(X)| \geq 0.$

Theorem 3 (Ishida-Matsuo-N)

If $M\#Z$ admits a solution of the normalized Ricci flow for all times $t \geq 0$ & $\exists C$, $\forall t \in [0,\infty)$, $|s_{g_t}| < C$,

$$\Rightarrow \qquad 4-(2\chi(Z)+3\tau(Z)) \leq \frac{1}{3}(2\chi(M)+3\tau(M)).$$

Ingredients of the proof of Theorem 2

Chern-Gauss-Bonnet + Hirzebruch

$$\frac{1}{4\pi^2}\int_X \left(\frac{s_g^2}{24} + 2|W^+| - \frac{|\overset{\circ}{r}|^2}{2} \right) d\mu_g = 2\chi(X) + 3\tau(X),$$

where W^+: positive Weyl curvature,
$\overset{\circ}{r}$: the trace free part of the Ricci curvature.

[Fact] g: Einstein $\Rightarrow \overset{\circ}{r} = 0$

Basic estimate 2

$$\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0 \quad \Rightarrow \quad \frac{1}{4\pi^2}\int_X \left(\frac{s_g^2}{24} + 2|W^+| \right) d\mu_g \geq \frac{2}{3}(a_+)^2.$$

Ingredients of the proof of Theorem 3

Instead of the relation $\overset{\circ}{r} = 0$ in the case of Einstein, we have

[Fang-Zhang-Zhang,'08]

If X admits a solution of the normalized Ricci flow for all times $t \geq 0$ & $\exists C, c > 0$, $\forall t \in [0, \infty)$, $|s_{g_t}| < C$, $\max_{x \in X} s_{g_t}(x) < -c$,

$$\int_0^\infty \int_X |\overset{\circ}{r}|^2 d\mu_g dt < \infty.$$

In particular,

$$\lim_{T \to \infty} \int_T^{T+1} \int_X |\overset{\circ}{r}|^2 d\mu_g dt = 0.$$

On the Non-vanishing theorem

M: compact Kähler,
$Z = Z_1 \# \cdots Z_k$ s.t $Z_i = \overline{\mathbb{C}\mathrm{P}}^2$, $S^1 \times \Sigma$, $S^1 \times Y$ ($g(\Sigma) \geq 1$, $\mathcal{Y}(Y) \geq 0$)

Non-vanishing theorem

$\exists\, \mathrm{Spin}^{c-}$-structure on $M \# Z$ s.t.

- $\mathrm{SW}^{\mathrm{Pin}^-(2)} \not\equiv 0$
- $\forall g \quad (a_+)^2 \geq c_1^2(M)$
  where a_+ is the g-self-dual part of the g-harmonic form a representing $\tilde{c}_1(E)$.

If $\exists Z_i = S^2 \times \Sigma$ or $S^1 \times Y$ with $b_1(Y) \geq 1$
$\Rightarrow$ SW & Donaldson inv. of $M \# Z$ are 0 ($\because b_+(M),\ b_+(Z) \geq 1$.)

In general,

[Fact]

If $b_+(X)$, $b_+(Y) \geq 1$,
$\Rightarrow$ all of Donaldson inv & SW inv of $X \# Y$ are 0.

However, if $b_+(Y) = 0$, SW can be nontrivial.

[Fintushel-Stern, Kotschick-Morgan-Taubes, Ozsbath-Szabo, Froyshov]

- $Y_1, \ldots, Y_k$: $b_1(Y_i) = b_+(Y_i) = 0$ or $Y_i = S^1 \times S^3$.
- X: $\mathrm{SW}(X) \neq 0$

$\Rightarrow \mathrm{SW}(X \# Y_1 \# \cdots \# Y_k) \neq 0$.

- $\mathrm{Pin}^{-}(2)$-monopole theory = SW theory twisted along the local coefficient l associated with the Spin^{c-}-structure.
- It can occur that $b_{+}^{l} = \dim H_{+}(X; l) = 0$, even if $b_{+} \neq 0$.
- For $Z_i = S^2 \times \Sigma$ or $S^1 \times Y$, $\exists\, \mathrm{Spin}^{c-}$-structure s.t. $b_{+}^{l} = 0$.

$\Rightarrow$ We can prove $\mathrm{SW}^{\mathrm{Pin}^{-}(2)}(M \# Z) \neq 0$.