$\mathrm{Pin}^{-}(2)$-monopole equations and applications

Nobuhiro Nakamura

Gakushuin university

Dec 20, 2012

Two worlds of manifolds

$$\mathrm{TOP_n} := \{n\text{-dimensional topological manifolds}\}/\text{homeo.}$$
$$\mathrm{DIFF_n} := \{n\text{-dimensional smooth manifolds})\}/\text{diffeo.}$$

We will assume every manifold is connected, closed and oriented.

Forgetful map

$$\varphi_n \colon \mathrm{DIFF_n} \to \mathrm{TOP_n}.$$

Basic question: Is φ_n injective, surjective?

- φ_n is NOT surjective $\Leftrightarrow$ $\exists$Nonsmoothable n-manifolds
- φ_n is NOT injective $\Leftrightarrow$ $\exists$Exotic smooth structures

- $n \leq 3 \Rightarrow \mathrm{TOP}_n = \mathrm{DIFF}_n$.
- $n \geq 4 \Rightarrow$ In general, $\mathrm{TOP}_n \neq \mathrm{DIFF}_n$.
  - $n \geq 5 \Rightarrow$ Algebraic topology (Surgery theory).
  - $n = 4 \Rightarrow$ Difficult
    - Freedman's theory $\longrightarrow \mathrm{TOP}_4$
    - Gauge theory $\longrightarrow \mathrm{DIFF}_4$

We concentrate on $n = 4$ below.

- Let X be a closed oriented 4-manifold.

Intersection form

$$Q_X \colon H^2(X;\mathbb{Z})/\text{torsion} \times H^2(X;\mathbb{Z})/\text{torsion} \to \mathbb{Z},$$
$$(a,b) \mapsto \langle a \cup b, [X] \rangle.$$

- Q_X is a symmetric bilinear unimodular form.

[J.H.C.Whitehead '49]

If $\pi_1 X = 1$, the homotopy type of X is determined by the isomorphism class of Q_X.

Some definitions

- Q_X: even $\Leftrightarrow \forall a$, $Q_X(a,a) \equiv 0$ mod 2.
- Q_X: odd $\Leftrightarrow \exists a$, $Q_X(a,a) \equiv 1$ mod 2.
- Q_X: positive (negative) definite $\Leftrightarrow \forall a \neq 0$, $Q_X(a,a) > 0$ (< 0).
- $b_+(X)$, $b_-(X)$: Decompose $H_2(X;\mathbb{Q}) = H_+(X) \oplus H_-(X)$, s.t. $Q_X \otimes \mathbb{Q}$ is posi.(nega.) definite on $H_+(X)$ ($H_-(X)$). Let $b_\pm(X) = \operatorname{rank} H_\pm(X)$
- $\operatorname{sign}(X) = b_+(X) - b_-(X)$.

Fact

- Q: even form $\Rightarrow \operatorname{sign} \equiv 0 \mod 8$.

Examples

- $X = S^4$, $Q_X = 0$.
- $X = S^2 \times S^2$, $Q_X = H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
  In general, $X = \Sigma_{g_1} \times \Sigma_{g_2}$, $Q_X = H \oplus \cdots \oplus H$, where Σ_g is a closed Riemann surface of genus g.
- $X = \mathbb{C}\mathrm{P}^2$, $Q_X = (1)$.
- $X = \overline{\mathbb{C}\mathrm{P}^2}$, $Q_X = (-1)$.
- $X = m\,\mathbb{C}\mathrm{P}^2 \# n\overline{\mathbb{C}\mathrm{P}}^2$,
  $$Q_X = \underbrace{(1) \oplus \cdots \oplus (1)}_{m} \oplus \underbrace{(-1) \oplus \cdots \oplus (-1)}_{n}.$$
- $X = K3$, $Q_X = 2(-E_8) \oplus 3H$.

In TOP_4

$\underline{\pi_1 X = 1}$

[Freedman '82]

The homeo type of X is determined by

- the iso. class of Q_X if Q_X is even,
- the iso. class of Q_X & $\mathrm{ks}(X)$ if Q_X is odd.

$\underline{\pi_1 X \neq 1}$

If $\pi_1 X$ is "Good" $\Rightarrow$ Freedman theory $+$ Surgery theory.
$\rightarrow$ Difficult.

In DIFF_4

[Rohlin,'52] If X is spin $\Rightarrow \operatorname{sign}(X) \equiv 0 \bmod 16$.

[Donaldson,'82] If Q_X is definite $\Rightarrow Q_X \sim$ The diagonal form.

[Furuta,'95] If X is spin & Q_X is indefinite, then

$$b_2(X) \geq \frac{10}{8}|\operatorname{sign}(X)| + 2.$$

- In these results, $\pi_1(X)$ is arbitrary.
- These results + Freedman's theory
  $\Rightarrow \exists$Many nonsmooothable 4-manifolds.

Refinements, variants when $\pi_1 \neq 1$

[Furuta-Kametani '05]

The strong $10/8$-inequality in the case when $b_1(X) > 0$.

[Froyshov '10]

A local coefficient analogue of Donaldson's theorem.

local coefficients $\leftrightarrow$ double coverings $\leftrightarrow$ $H^1(X;\mathbb{Z}/2)$

Froyshov's results

4-manifolds and intersection forms with local coefficients, arXiv:1004.0077

- Suppose a double covering $\tilde{X} \to X$ is given.
- $l := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, a $\mathbb{Z}$-bundle over X.
  $\longrightarrow H^*(X; l)$: l-coefficient cohomology.
- Note $l \otimes l = \mathbb{Z}$. The cup product

  $$\cup \colon H^2(X; l) \times H^2(X; l) \to H^4(X; \mathbb{Z}) \cong \mathbb{Z},$$

  induces the intersection form with local coefficient

  $$Q_{X,l} \colon H^2(X; l)/\text{torsion} \times H^2(X; l)/\text{torsion} \to \mathbb{Z}.$$

- $Q_{X,l}$ is also a symmetric bilinear unimodular form.

Example

- V: closed 4-manifold with $\pi_1 V = 1$.
- $X = V\#(S^2 \times T^2)$.
- l: a nontrivial $\mathbb{Z}$-bundle over X.

$$Q_X = Q_V \oplus H, \qquad Q_{X,l} = Q_V.$$

- E: Enriques surface $\Rightarrow \pi_1 E = \mathbb{Z}/2$
- l: a nontrivial $\mathbb{Z}$-bundle over E.

$$Q_E = (-E_8) \oplus H, \qquad Q_{E,l} = (-E_8) \oplus H \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

A special case of Froyshov's theorem

- X: a closed connected oriented smooth 4-manifold s.t.

$$b^+(X) + \dim_{\mathbb{Z}/2}(\mathrm{tor} H_1(X;\mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2. \qquad (1)$$

- $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

- The original form of Froyshov's theorem is:

  If X with $\partial X = Y : \mathbb{Z}HS^3$ satisfies (1)
  & $Q_{X,l}$ is nonstandard definite
  $\Rightarrow \delta_0 \colon HF^4(Y;\mathbb{Z}/2) \to \mathbb{Z}/2$ is non-zero.
- $Y = S^3 \Rightarrow HF^4(Y;\mathbb{Z}/2) = 0 \Rightarrow$ The above result.

- The proof uses $\mathrm{SO}(3)$-Yang-Mills theory on a $\mathrm{SO}(3)$-bundle V.
- Twisted reducibles (stabilizer $\cong \mathbb{Z}/2$) play an important role.
  V is reduced to $\lambda \oplus E$, where E is an $\mathrm{O}(2)$-bundle,
  $\lambda = \det E$: a nontrivial $\mathbb{R}$-bundle.

Cf [Fintushel-Stern'84]'s proof of Donaldson's theorem also used $\mathrm{SO}(3)$-Yang-Mills.
$\longrightarrow$ Abelian reducibles (stabilizer $\cong \mathrm{U}(1)$)
V is reduced to $\underline{\mathbb{R}} \oplus L$, where L is a $\mathrm{U}(1)$-bundle.

- Donaldson's theorem is proved by Seiberg-Witten theory, too.

Question

Can we prove Froyshov's result by Seiberg-Witten theory?

$\longrightarrow$ Our result would be an answer.

Theorem 1.(N.)

- X: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

$$\boxed{\text{If } Q_{X,l} \text{ is definite} \Rightarrow Q_{X,l} \sim \text{diagonal.}}$$

Cf. Froyshov's theorem

- X: — s.t. $b^+(X) + \dim_{\mathbb{Z}/2}(\mathrm{tor} H_1(X;\mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2$.
- $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

$$\boxed{\text{If } Q_{X,l} \text{ is definite} \Rightarrow Q_{X,l} \sim \text{diagonal.}}$$

Theorem 1.(N.)

- X: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow$ $Q_{X,l} \sim$ diagonal.

- For the proof, a variant of Seiberg-Witten ($\mathrm{U}(1)$-monopole) equations is introduced

$\longrightarrow$ $\mathrm{Pin}^-(2)$-monopole equations

- $\mathrm{Pin}^-(2)$-monopole eqns are defined on a Spin^{c-}-structure.
- Spin^{c-}-structure is a $\mathrm{Pin}^-(2)$-variant of Spin^c-str. defined by M.Furuta, whose complex structure is "twisted along l".

- The moduli space of $\mathrm{Pin}^-(2)$-monopoles is compact.
  $\longrightarrow$ Bauer-Furuta theory can be developed.

Furuta's theorem

Let X be a closed ori. smooth spin 4-manifold with indefinite Q_X.

$$b_2(X) \geq \frac{10}{8}|\operatorname{sign}(X)| + 2.$$

Theorem 2(N.)

Let X be a closed connected ori. smooth 4-manifold. For any nontrivial $\mathbb{Z}$-bundle $l \to X$ s.t. $w_1(\lambda)^2 = w_2(X)$, where $\lambda = l \otimes \mathbb{R}$,

$$b_2(X; l) \geq \frac{10}{8}|\operatorname{sign}(X)|,$$

where $b_2(X; l) = \operatorname{rank} H_2(X; \lambda)$.

A new class of nonsmoothable 4-manifolds

Recall fundamental theorems.

1. [Rohlin] X^4: closed spin $\Rightarrow \operatorname{sign}(X) \equiv 0$ mod 16.
2. [Donaldson] Definite $\Rightarrow$ diagonal.
3. [Furuta] The 10/8-inequality

3' [Furuta-Kametani] The strong 10/8-inequality in the case when $b_1 > 0$.

Corollary 1(N.)

$\exists$ Nonsmoothable closed indefinite spin 4-manifolds satisfying

- $\operatorname{sign}(X) \equiv 0$ mod 16,
- the strong 10/8-inequality.

Proof

- Let M be T^4 or $T^2 \times S^2$. $\Rightarrow Q_{T^4} = 3H$, $Q_{T^2 \times S^2} = H$.
- If $l' \to M$ is any nontrivial $\mathbb{Z}$-bundle, $\Rightarrow b_2(M; l') = 0$ & $w_1(l' \otimes \mathbb{R})^2 = 0$.
- Let V be a topological 4-manifold s.t. $\pi_1 V = 1$, Q_V is even and definite, $\operatorname{sign}(V) \equiv 0 \bmod 16$. ($\Rightarrow V$ is spin.)
- Choose a large k s.t. $X = V \# kM$ satisfies the strong 10/8-inequality.
- Let $l := \underline{\mathbb{Z}} \# k l' \to X$. $\Rightarrow Q_{X,l} = Q_V$, $w_1(l \otimes \mathbb{R})^2 = 0$.
- Suppose X is smooth. By Theorem 1, $Q_{X,l} = Q_V \sim$ diagonal. Contradiction.

$\mathrm{Spin}^{c-}(4)$

$$\mathrm{Pin}^-(2) = \langle \mathrm{U}(1), j \rangle = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}.$$

Two-to-one homomorphism $\mathrm{Pin}^-(2) \to \mathrm{O}(2)$

$$z \in \mathrm{U}(1) \subset \mathrm{Pin}^-(2) \mapsto z^2 \in \mathrm{U}(1) \cong \mathrm{SO}(2) \subset \mathrm{O}(2),$$
$$j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition $\mathrm{Spin}^{c-}(4) := \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$.

- $\mathrm{Spin}^{c-}(4)/\mathrm{Pin}^-(2) = \mathrm{Spin}(4)/\{\pm 1\} = \mathrm{SO}(4)$
- $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}(4) = \mathrm{O}(2)$
- The id. compo. of $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$
  $= \mathrm{Spin}^c(4)$

  $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}^c(4) = \{\pm 1\}$.

Spin^{c-}-structures

- X: an oriented Riemannian 4-manifold.
  $\longrightarrow Fr(X)$: The $\mathrm{SO}(4)$-frame bundle.
- $\tilde{X} \overset{2:1}{\to} X$: nontrivial double covering, $l := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$

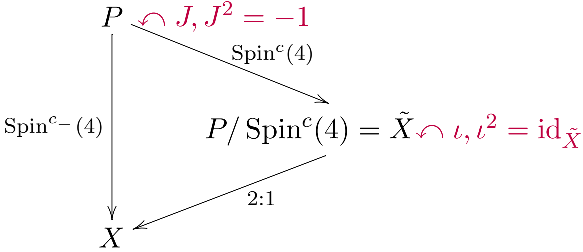
Spin^{c-}-structure

A Spin^{c-}-structure on $\tilde{X} \to X$ is given by

- P: a $\mathrm{Spin}^{c-}(4)$-bundle over X,
- $P/\mathrm{Spin}^c(4) \overset{\cong}{\to} \tilde{X}$
- $P/\mathrm{Pin}^-(2) \overset{\cong}{\to} Fr(X)$.

Then we have

- $E = P/\mathrm{Spin}(4) \overset{\mathrm{O}(2)}{\to} X$: characteristic $\mathrm{O}(2)$-bundle.
  $\longrightarrow$ l-coefficient Euler class $\tilde{c}_1(E) \in H^2(X; l)$.
  $H^2(X; l) \overset{1:1}{\longleftrightarrow} \{\mathrm{O}(2)\text{-bundle } E \text{ over } X \text{ s.t. } E/\mathrm{SO}(2) \cong \tilde{X}\}/\text{iso}$.



- $P \stackrel{\mathrm{Spin}^c(4)}{\longrightarrow} \tilde{X}$ defines a Spin^c-structure on $\tilde{X}$
- $J = [1, j] \in \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2) = \mathrm{Spin}^{c-}(4)$.

A Spin^{c-}-str. on X can be given by the data

- A Spin^c-structure on $\tilde{X}$

$$(P_c \overset{\mathrm{Spin}^c(4)}{\longrightarrow} \tilde{X},\ P_c/\,\mathrm{U}(1) \overset{\cong}{\to} Fr(\tilde{X}))$$

- A fiber preserving diffeo. $J\colon P_c \to P_c$ covering $\iota\colon \tilde{X} \to \tilde{X}$ s.t.
  - $J^2 = -1$
  - $J(pg) = J(p)\bar{g}$, where

    $$\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1) \ni g = [q, z] \mapsto \bar{g} = [q, z^{-1}]$$

    J is NOT a $\mathrm{Spin}^c(4)$-bundle auto.
  - J induces $\iota_*\colon Fr(\tilde{X}) \to Fr(\tilde{X})$

Define the action I on the spinor bundles:

$$\tilde{S}^{\pm} = P_c \times_{\mathrm{Spin}^c(4)} \mathbb{H}_{\pm} \curvearrowleft [J, j] =: I$$

$\Rightarrow I^2 = 1$ & I is antilinear.
$\Rightarrow S^{\pm} = \tilde{S}^{\pm}/I$ are the spinor bundles for the Spin^{c-}-str.
$S^{\pm}$ are not complex bundles.

I induces an antilinear involution on $L = \det \tilde{S}^{+}$.
$\Rightarrow$ The characteristic $\mathrm{O}(2)$-bundle $E = L/I$.

$\mathrm{Pin}^{-}(2)$-monopole on X = I-invariant Seiberg-Witten on $\tilde{X}$

In fact, $\mathcal{M}_{\mathrm{Pin}} = (\mathcal{M}_{SW})^I$

Take the I-invariant part of the monopole map μ_{SW} on $\tilde{X}$.
$\Rightarrow \mathrm{Pin}^-(2)$-monopole map,

$$\mu\colon \mathcal{A} \times \Gamma(S^+) \to i\Omega^+(l \otimes \mathbb{R}) \times \Gamma(S^-)$$

where
$\mathcal{A} = \{\mathrm{O}(2)\text{-connections on } E\}$ ← an affine sp. of $i\Omega^1(l \otimes \mathbb{R})$

Gauge symmetry

$$\begin{aligned}\mathcal{G} =& \{f \in \mathrm{Map}(\tilde{X}, \mathrm{U}(1)) \mid f(\iota x) = f(x)^{-1}\} \\ =& \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1)),\end{aligned}$$

where $\{\pm 1\} \curvearrowright \mathrm{U}(1)$ by $z \mapsto z^{-1}$.

Moduli spaces

$$\mathcal{M} = \mu^{-1}(0)/\mathcal{G} \subset (\mathcal{A} \times \Gamma(S^+))/\mathcal{G}$$

Proposition

- $\mathcal{M}$ is compact.
- The virtual dimension of $\mathcal{M}$:

$$d = \frac{1}{4}(\tilde{c}_1(E)^2 - \operatorname{sign}(X)) - (b_0^l - b_1^l + b_+^l).$$

where $b_\bullet^l = \operatorname{rank} H_\bullet(X; l)$.

- If l is nontrivial & X connected $\Rightarrow b_0^l = 0$.

Reducibles

- For $(A, \Phi) \in \mathcal{A} \times \Gamma(S^+)$, if $\Phi \not\equiv 0 \Rightarrow \mathcal{G}$-action is free.
- The stabilizer of $(A, \Phi \equiv 0) = \{\pm 1\}$
- The elements of the form $(A, \Phi \equiv 0)$ are called reducibles.
- In general, $\{$ reducible solutions $\}/\mathcal{G} \cong T^{b_1^l} \subset \mathcal{M}$.

Cf. In the SW-case, the stabilizer of $(A, 0) = S^1 \subset \mathrm{Map}(X, S^1)$.

Key difference

Ordinary SW case

- Reducible $\rightarrow$ The stabilizer $= S^1$.

$$\mathcal{M}_{SW} \setminus \{\text{reducibles}\} \subset (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G}_{SW} \simeq B\mathcal{G}_{SW}$$
$$\simeq T^{b_1} \times \mathbb{C}\mathrm{P}^{\infty}.$$

$\mathrm{Pin}^{-}(2)$-monopole case

- Reducible $\rightarrow$ The stabilizer $= \{\pm 1\}$.

$$\mathcal{M} \setminus \{\text{reducibles}\} \subset (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G} \simeq B\mathcal{G}$$
$$\simeq T^{b_1^l} \times \mathbb{R}\mathrm{P}^{\infty}.$$

Proof of Theorem 1

- For simplicity, assume $b_1^l = 0$.

Lemma 1

$\forall$ characteristic elements w of $Q_{X,l}$,

$$|w^2| \geq b_2^l.$$

An element w in a unimodular lattice L is called *characteristic* if $w \cdot v \equiv v \cdot v$ mod 2 for $\forall v \in L$.

[Elkies '95]

$L \subset \mathbb{R}^n$: unimodular lattice. If $\forall$characteristic element $w \in L$ satisfies $|w^2| \geq \operatorname{rank} L$, $\Rightarrow L \cong$ diagonal.

Lemma 1 & [Elkies '95] $\Rightarrow Q_{X,l} \sim$ diagonal.

Lemma 2

If $w_1(l \otimes \mathbb{R})^2 = 0$

$\Rightarrow \forall$ characteristic element w, $\exists\,\mathrm{Spin}^{c-}$-str. s.t. $\tilde{c}_1(E) = w$.

Lemma 3

If $b^l_+ = b^l_1 = 0 \Rightarrow \dim \mathcal{M} \leq 0$ for $\forall\,\mathrm{Spin}^{c-}$-str.

Lemma 2 & 3 $\Rightarrow$ Lemma 1

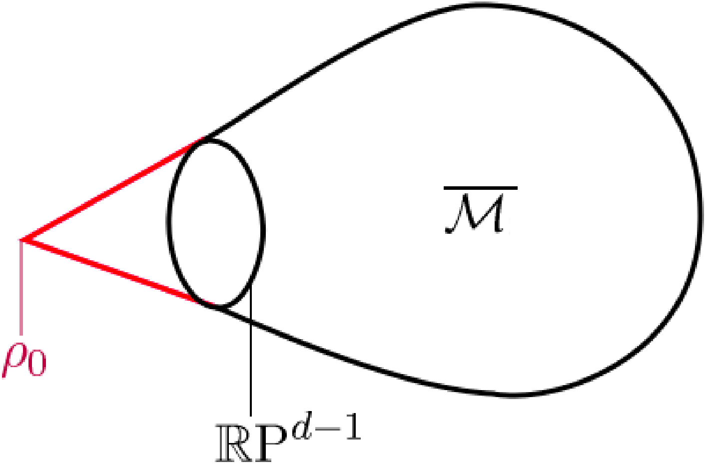
$$0 \geq \dim \mathcal{M} = \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b^l_0 - b^l_1 + b^l_+) = \frac{1}{4}(-|w^2| + b^l_2)$$

The structure of $\mathcal{M}$ when $b_+(X;l) = 0$

- Suppose a Spin^{c-}-structure (P, τ) on X is given.
- $b_1(X, l) = 0 \Rightarrow \exists^1$ reducible class $\rho_0 \in \mathcal{M}$.
- Perturb the $\mathrm{Pin}^-(2)$-monopole equations by adding $\eta \in \Omega^+(i\lambda)$ to the curvature equation. $\to F_A^+ = q(\phi) + \eta$.
- For generic η, $\mathcal{M} \setminus \{\rho_0\}$ is a d-dimensional manifold.
- Fix a small neighborhood $N(\rho_0)$ of $\{\rho_0\}$.

  $\Rightarrow N(\rho_0) \cong \mathbb{R}^d/\{\pm 1\} =$ a cone of $\mathbb{R}\mathrm{P}^{d-1}$

  Then $\overline{\mathcal{M}} := \overline{\mathcal{M} \setminus N(\rho_0)}$ is a compact d-manifold & $\partial\overline{\mathcal{M}} = \mathbb{R}\mathrm{P}^{d-1}$.


$\overline{\mathcal{M}}$
ρ_0
$\mathbb{R}P^{d-1}$

- Note $\overline{\mathcal{M}} \subset (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G} =: \mathcal{B}^*$.
- Recall $\mathcal{B}^* \underset{h.e.}{\simeq} T^{b_1(X;l)} \times \mathbb{R}\mathrm{P}^\infty$.

Lemma 3

If $b_+^l = 0$ & $b_1^l = 0 \Rightarrow d = \dim \mathcal{M} \leq 0$.

Proof

- Suppose $d > 0$.
- Recall $\overline{\mathcal{M}}$ is a compact d-manifold s.t. $\partial\overline{\mathcal{M}} = \mathbb{R}\mathrm{P}^{d-1}$.
- $\exists C \in H^{d-1}(\mathcal{B}^*; \mathbb{Z}/2) \cong H^{d-1}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$ s.t. $\langle C, [\partial\overline{\mathcal{M}}]\rangle \neq 0$. $\Rightarrow$ Contradiction.

The outline of the proof of Theorem 2

- If $E = \underline{\mathbb{R}} \oplus \lambda \Rightarrow \mathrm{Spin}^{c-}$-structure on (X, E) has the larger symmetry $\mathcal{G}' = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{Pin}^-(2))$.
- For simplicity, assume $b_1(X; l) = 0$.
- Then, by taking finite dimensional approximation of the monopole map, we obtain a proper $\mathbb{Z}_4$-equivariant map

$$f \colon \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

  where
  - $\tilde{\mathbb{R}}$ is $\mathbb{R}$ on which $\mathbb{Z}_4$ acts via $\mathbb{Z}_4 \to \mathbb{Z}_2 = \{\pm 1\} \curvearrowright \mathbb{R}$,
  - $\mathbb{C}_1$ is $\mathbb{C}$ on which $\mathbb{Z}_4$ acts by multiplication of i,
  - $k = -\operatorname{sign}(X)/8$, $b = b_+(X; \lambda)$, m, n are some integers.

Here, $\mathbb{Z}_4$ is generated by the constant section

$$j \in \mathcal{G}' = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{Pin}^-(2)).$$

- By using the techniques of equivariant homotopy theory, e.g., tom Dieck's character formula, we can see that any proper $\mathbb{Z}_4$-map of the form,

$$f\colon \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

  should satisfy $b \geq k$.
- That is,

$$b_+(X;\lambda) \geq -\frac{1}{8}\operatorname{sign}(X).$$

Finite dimensional approximation

- Take a flat connection A_0 on $\underline{\mathbb{R}} \oplus \lambda$.

$\mathrm{Pin}^-(2)$-monopole map

$$\mu\colon \Omega^1(i\lambda) \oplus \Gamma(S^+) \to (\Omega^0 \oplus \Omega^+)(i\lambda) \oplus \Gamma(S^-) =: \mathcal{W},$$
$$(a, \phi) \mapsto (d^* a, F_{A_0} + d^+ a + q(\phi), D_{A_0+a}\phi).$$

- Let $l(a, \phi) := (d^* a, d^+ a, D_{A_0}\phi)$ be the linear part of μ.
  $\to l$ is Fredholm.
- $c = \mu - l$: quadratic, compact.
- Choose a finite dim. subspace $U \subset \mathcal{W}$ s.t. $\dim U \gg 1$,
  $U \supset (\operatorname{im} l)^\perp$
- Let $V := l^{-1}(U)$ & $p\colon \mathcal{W} \to U$ be the L^2-projection.
- Define $f\colon V \to U$ by $\boxed{f = l + pc.}$ $\to f$: proper, $\mathbb{Z}_4$-equiv.

$\mathrm{Pin}^{-}(2)$-monopole invariants

If $b^l_+ \geq 1 \Rightarrow$ by perturbing the eqns

- $\mathcal{M}$ contains no reducible.
- $\mathcal{M}$ is a finite dimensional manifold.

$$\mathcal{M} \subset (\mathcal{A}_{\mathrm{O}(2)} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G} \underset{h.e}{\simeq} T^{b^l_1} \times \mathbb{R}\mathrm{P}^\infty$$

$H^*(\mathbb{R}\mathrm{P}^\infty \times T^{b^l_1}; \mathbb{Z}_2) = \mathbb{Z}_2[\eta] \otimes \bigwedge \mathbb{Z}_2^{b^l_1}$, where η: generator of $H^1(\mathbb{R}\mathrm{P}^\infty)$.

$\mathrm{Pin}^{-}(2)$-monopole invariants

Define $\mathrm{SW}^{\mathrm{Pin}} \colon \mathbb{Z}_2[\eta] \otimes \bigwedge \mathbb{Z}_2^{b^l_1} \to \mathbb{Z}_2$ by

$$\mathrm{SW}^{\mathrm{Pin}}(\eta^k \otimes t) = \langle \eta^k \otimes t, [\mathcal{M}] \rangle.$$

If $b^l_+ \geq 2 \Rightarrow \mathrm{SW}^{\mathrm{Pin}}$ is a diffeomorphism invariant of X.

Exotic structures

- $E(1) = \mathbb{C}\mathrm{P}^2 \# 9\overline{\mathbb{C}\mathrm{P}}^2$.
- $E(n) = \underbrace{E(1)\#_f \cdots \#_f E(1)}_{n}$. Note $E(2) = K3$.

Theorem(due to many people)

$\exists\infty$ many exotic smooth structures on $E(n)$ for $n \geq 1$.

- Construction:
  - Log transform.
  - Knot surgery ([Fintushel-Stern,'98])
- Method to detect exotic structures
  - Donaldson invariants $\longleftarrow$ ASD equation
  - Seiberg-Witten invariants $\longleftarrow$ SW equations

Exotic small manifolds

[Kotchick,'89] Let B be the Barlow surface.

$$B \underset{\text{homeo.}}{\cong} \mathbb{C}\mathrm{P}^2 \# 8\overline{\mathbb{C}\mathrm{P}}^2 \text{ but } B \underset{\text{diffeo.}}{\not\cong} \mathbb{C}\mathrm{P}^2 \# 8\overline{\mathbb{C}\mathrm{P}}^2.$$

- Method: Donaldson inv.

[J.Park,'03] $\exists$ exotic structures on $\mathbb{C}\mathrm{P}^2 \# 7\overline{\mathbb{C}\mathrm{P}}^2$.

- Construction: Rational blow down [Fintushel-Stern,'97]
- Method: SW inv.

At present, it is known
$\exists$ exotic structures on $\mathbb{C}\mathrm{P}^2 \# 2\overline{\mathbb{C}\mathrm{P}}^2$.

Stabilization & Vanishing results

[Wall,'64]

$\pi_1 X_1 = \pi_1 X_2 = 1$ & $Q_{X_1} \cong Q_{X_2}$ $\Rightarrow X_1 \underset{\text{homeo.}}{\cong} X_2$. Possibly $X_1 \underset{\text{diffeo.}}{\not\cong} X_2$

$$\Rightarrow \ \exists k \text{ s.t. } X_1 \# k(S^2 \times S^2) \underset{\text{diffeo.}}{\cong} X_2 \# k(S^2 \times S^2).$$

Theorem ([Donaldson, Witten])

- X_1, X_2: $b_+(X_1) > 0$, $b_+(X_2) > 0$.

Then all of Donaldson invariants & Seiberg-Witten invariants of $X_1 \# X_2$ are 0.

X_1, X_2: exotic pair, Y with $b_+(Y) > 0$
$\Rightarrow$ Cannot prove the pair $X_1 \# Y$ and $X_2 \# Y$ is an exotic pair by using Donaldson & SW inv.

On the other hand,

[Fintushel-Stern, Kotschick-Morgan-Taubes, Froyshov]

- $Y_1, \dots, Y_k$: $b_1(Y_i) = b_+(Y_i) = 0$ e.g., $Y_i = \overline{\mathbb{C}\mathrm{P}}^2$, $\mathbb{Q}HS^4$ or $Y_i = S^1 \times S^3$.
- X: $\mathrm{SW}(X) \neq 0$

$\Rightarrow \mathrm{SW}(X \# Y_1 \# \cdots \# Y_k) \neq 0$.

By using this, we can prove
$\exists$exotic str. on $E(n) \# Y_1 \# \cdots \# Y_k$.

By using $\mathrm{SW}^{\mathrm{Pin}}$, we can obtain

Theorem (N.)

For $\forall k$, $g_i \geq 1$ $(1 \leq i \leq k)$,

$\exists\infty$ many exotic structures on $E(n)\#(S^2 \times \Sigma_{g_1})\# \cdots \#(S^2 \times \Sigma_{g_k})$.

- $\mathrm{Pin}^-(2)$-monopole = SW twisted along a local coefficient.
- For some local coefficient l,

$$b_+^l(S^2 \times \Sigma_g) := \dim H_+(S^2 \times \Sigma_g; l) = 0.$$

$$\Rightarrow \mathrm{SW}^{\mathrm{Pin}}(E(n)\#(S^2 \times \Sigma_{g_1})\# \cdots \#(S^2 \times \Sigma_{g_k})) \neq 0.$$

Theorem (N.)

- X: 4-manifold, for a Spin^c-str. c_1, SW-inv is odd.
- Y: 4-manifold with nontrivial double covering $\tilde{Y}$, s.t.
  - $\exists$ positive scalar curvature metric
  - $\exists$ Spin^{c-}-str. c_2
    s.t. $b^l_+ = 0$ & v-dim $\mathcal{M} = b^l_1$, for $l = \tilde{Y} \times_{\{\pm 1\}} \mathbb{Z}$.
    $\Rightarrow \mathcal{M} = \{$reducibles only$\}/\mathcal{G} \cong T^{b^l_1}$ & transversal

For $c_1 \# c_2$ over $X \# Y$,

$$\mathrm{SW}^{\mathrm{Pin}}(\eta \otimes t^{\mathsf{top}}) \neq 0,$$

where t^{top} is the generator of $H^{b^l_1}(T^{b^l_1})$.

- The virtual dimension of $\mathcal{M}(X \# Y, c_1 \# c_2)$ is positive.
- The ordinary SW & stable cohomotopy SW of $X \# Y$ are 0.
  ($\because Y$ admits a PSC metric.)

More exotic connected sums

By using stable cohomotopy SW invariants ([Bauer-Furuta]),

- $\underbrace{K3\#\cdots\#K3}_{k}$: $1 \leq k \leq 4 \Rightarrow \exists$exotic
- $X = E(n_1)\#E(n_2)\#E(n_3)\#E(n_4)$

  n_i: even, and $b_+(X) \underset{(8)}{\equiv} 4 \Rightarrow \exists$exotic
- [Sasahira] $M_1, M_2 = K3$ or $\Sigma_g \times \Sigma_{g'}$ (g, g': odd)

  $\Rightarrow \exists$exotic str. on $K3\#M_1$ and $K3\#M_1\#M_2$.

The genus of embedded surfaces in 4-manifolds

Theorem

- X: closed ori. 4-manifold with $b_+ \geq 2$.
- c: Spin^c-structure on X.
  L: the determinant line bundle of c.
- $\Sigma \subset X$: connected embedded surface
  s.t. $[\Sigma] \in H_2(X;\mathbb{Z})$, $[\Sigma]\cdot[\Sigma] \geq 0$.

If $\mathrm{SW}(X,c) \neq 0$ or stable cohomotopy $\mathrm{SW}(X,c) \neq 0$, then

$$-\chi(\Sigma) = 2g - 2 \geq |c_1(L)[\Sigma]| + [\Sigma]\cdot[\Sigma].$$

- This is due to: [Kronheimer-Mrowka], [Fintushel-Stern], [Morgan-Szabo-Taubes], [Ozsvath-Szabo], [Furuta-Kametani-Matsue-Minami]...

Embedded surfaces representing a class in $H_2(X; l)$

- $\tilde{X} \to X$: nontrivial double covering, $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$.

Let us consider a connected surface Σ s.t.

- $i \colon \Sigma \hookrightarrow X$: embedding
- (The orientation coefficient of Σ) $= i^* l$

$\to$ $\exists$Fundamental class $[\Sigma] \in H_2(\Sigma; i^* l)$.
Let $\alpha := i_*[\Sigma] \in H_2(X; l)$, where $i_* \colon H_2(\Sigma; i^* l) \to H_2(X; l)$.

Proposition

For $\forall \alpha \in H_2(X; l)$, there exists Σ as above.

Remark

- Σ may be orientable or nonorientable.

Theorem (N.)

- (X, l, Σ) as above. Suppose $b_+^l \geq 2$.
- Let $[\Sigma] \in H_2(X; l)$.
  Suppose $[\Sigma] \cdot [\Sigma] \geq 0$, & $[\Sigma]$ is not a torsion.
- c: Spin^{c-}-structure $\rightarrow$ The associated $\mathrm{O}(2)$-bundle E
- $\tilde{c}$: the Spin^c-structure on $\tilde{X}$ induced from c.

If one of the following is nonzero

- $\mathrm{SW}^{\mathrm{Pin}}$ or stable cohomotopy $\mathrm{SW}^{\mathrm{Pin}}$ of (X, c),
- SW or stable cohomotopy SW of $(\tilde{X}, \tilde{c})$,

then

$$-\chi(\Sigma) \geq [\Sigma] \cdot [\Sigma] + |\tilde{c}_1(E) \cdot [\Sigma]|.$$

Example

- $X = K3\#(S^2 \times \Sigma_1)\# \cdots \#(S^2 \times \Sigma_k)$, $(g_i \geq 1)$.
- $\exists l$ s.t. $H_2(X;l) = H_2(K3;\mathbb{Z}) \oplus$ Torsion.
- $\exists c$ s.t. $\tilde{c}_1(E) = 0$ & $\mathrm{SW}^{\mathrm{Pin}}(X,c) \neq 0$.

For $\Sigma \hookrightarrow X$ s.t. $[\Sigma] \in H_2(X;l)$ & $[\Sigma]\cdot[\Sigma] \geq 0$ & $[\Sigma]$ is not a torsion,

$$-\chi(\Sigma) \geq [\Sigma]\cdot[\Sigma].$$

Final remarks for future researches

- $\mathrm{Pin}^-(2)$-monopole invariants
  - Calculation, gluing formula, stable cohomotopy refinements
- $\mathrm{Pin}^-(2)$-monopole on branched coverings
  - Exotic involutions
  - Smooth inequivalent but topologically equivalent embedded surfaces *Cf.* [Fintushel-Stern-Snukujian], [H.J.Kim-Ruberman]
- When $\tilde{X}$: symplectic & $I^*\omega = -\omega$,

  $$\mathrm{Pin}^-(2)\text{-monopole inv.} \underset{??}{=} \text{real Gromov-Witten inv.}$$

  Cf. [Tian-Wang]
- $\mathrm{Pin}^-(2)$-monopole Floer theory?
  $\mathrm{Pin}^-(2)$ Heegaard Floer theory?
- "Witten conjecture" for $\mathrm{Pin}^-(2)$-monopole invariants?
  - [Feehan-Leness] SW = Donaldson

    $\mathrm{Pin}^-(2)$-monopole inv. = ???