$\mathrm{Pin}^{-}(2)$-monopole invariants and applications

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Introduction

- $E(n)$: elliptic surf. (fiber sum of $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$)
  e.g., $E(2) = K3$.

Fact $\exists\infty$ many exotic structures on $E(n)$.

- Construction of exotic structures
  - Log transformations
  - [Fintushel-Stern] Knot surgery
- To detect exotic structures
  - Donaldson invariants
  - Seiberg-Witten invariants

Main Theorem

Main Theorem (N.)

For $\forall k$, $g_i \geq 1$ $(1 \leq i \leq k)$,

$\exists\infty$ many exotic structures on $E(n)\#(S^2 \times \Sigma_{g_1})\#\cdots\#(S^2 \times \Sigma_{g_k})$.

- Construction of exotic structures
  - Log transformations, Knot surgery
- To detect exotic structures
  $\longrightarrow$ $\mathrm{Pin}^-(2)$-monopole invariants

Cf. [Wall]

Even if $E(n)'$ is an exotic $E(n)$,

$$\exists k, \quad E(n)'\#k(S^2 \times S^2) \underset{\text{diffeo.}}{\cong} E(n)\#k(S^2 \times S^2)$$

Connected sums & Exotic structures

In general, it might not be easy to find an exotic structures on connected sums, because

[Fact]

If $b_+(X_1)$, $b_+(X_2) \geq 1$,
$\Rightarrow$ all of Donaldson inv & SW inv of $X_1 \# X_2$ are 0.

e.g., D inv & SW inv of $E(n)\#(S^2 \times \Sigma_g)$ are 0.

[Fintushel-Stern, Kotschick-Morgan-Taubes, Froyshov]

- $Y_1, \ldots, Y_k$: $b_1(Y_i) = b_+(Y_i) = 0$ or $Y_i = S^1 \times S^3$.
- X: $\mathrm{SW}(X) \neq 0$

$\Rightarrow \mathrm{SW}(X \# Y_1 \# \cdots \# Y_k) \neq 0$.

$\Rightarrow$ $\exists$exotic str. on $E(n) \# Y_1 \# \cdots \# Y_k$.

Our Main theorem is an analogy of this.

- For some local coefficient l,

$$b_+^l(S^2 \times \Sigma_g) := \dim H_+(S^2 \times \Sigma_g; l) = 0.$$

- $\mathrm{Pin}^-(2)$-monopole = SW twisted along a local coefficient.

More exotic connected sums

By using stable cohomotopy SW invariants ([Bauer-Furuta]),

- $\underbrace{K3\#\cdots\#K3}_{k}$: $1 \leq k \leq 4 \Rightarrow \exists$exotic
- $X = E(n_1)\#E(n_2)\#E(n_3)\#E(n_4)$
  n_i: even, and $b_+(X) \underset{(8)}{\equiv} 4 \Rightarrow \exists$exotic
- [Sasahira] $M_1, M_2 = K3$ or $\Sigma_g \times \Sigma_{g'}$ (g, g': odd)
  $\Rightarrow \exists$exotic str. on $K3\#M_1$ and $K3\#M_1\#M_2$.

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$\mathrm{Pin}^-(2)$-monopole equations

Spin^{c-}-structure

- $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$,
  $\mathrm{Pin}^-(2) = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1)$
- $\mathrm{Spin}^{c-}(4)/\mathrm{Pin}^-(2) = \mathrm{Spin}(4)/\{\pm 1\} = \mathrm{SO}(4)$
- $\mathrm{Spin}^{c-}(4) \supset \mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$
  $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}^c(4) = \{\pm 1\}$.

Let X be a closed ori. Riemannian 4-manifold with (nontrivial) double covering $\tilde{X} \overset{2:1}{\to} X$

Definition

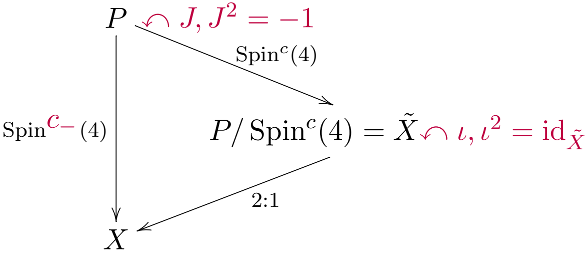
Spin^{c-}-structure on X is a $\mathrm{Spin}^{c-}(4)$-bundle P over X with

$$P/\mathrm{Pin}^-(2) \overset{\cong}{\to} Fr(X), \quad P/\mathrm{Spin}^c(4) \overset{\cong}{\to} \tilde{X}$$

Characteristic $\mathrm{O}(2)$-bundle

- $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}(4) = \mathrm{Pin}^-(2)/\{\pm 1\} = \mathrm{O}(2)$

$\Rightarrow E = P/\mathrm{Spin}(4)$ is an $\mathrm{O}(2)$-bundle
$\to$ Characteristic $\mathrm{O}(2)$-bundle



- $P \xrightarrow{\mathrm{Spin}^c(4)} \tilde{X}$ defines a Spin^c-structure on $\tilde{X}$
- $J = [1, j] \in \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2) = \mathrm{Spin}^{c-}(4)$.

A Spin^{c-}-str. on X is given by the data

- A Spin^c-structure on $\tilde{X}$

$$(P_c \overset{\mathrm{Spin}^c(4)}{\longrightarrow} \tilde{X},\ P_c/\,\mathrm{U}(1) \overset{\cong}{\to} Fr(\tilde{X}))$$

- A fiber preserving diffeo. $J\colon P_c \to P_c$ covering $\iota\colon \tilde{X} \to \tilde{X}$ s.t.
  - $J^2 = -1$
  - $J(pg) = J(p)\bar{g}$, where

    $$\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1) \ni g = [q, z] \mapsto \bar{g} = [q, z^{-1}]$$

    J is NOT a $\mathrm{Spin}^c(4)$-bundle auto.
  - J induces $\iota_*\colon Fr(\tilde{X}) \to Fr(\tilde{X})$

Define the action I on the spinor bundles:

$$\tilde{S}^{\pm} = P_c \times_{\mathrm{Spin}^c(4)} \mathbb{H}_{\pm} \curvearrowleft [J, j] =: I$$

$\Rightarrow$ $I^2 = 1$ & I is antilinear.
$\Rightarrow$ $S^{\pm} = \tilde{S}^{\pm}/I$ are the spinor bundles for the Spin^{c-}-str.
$S^{\pm}$ are not complex bundles.

I induces an antilinear involution on $L = \det \tilde{S}^{+}$.
$\Rightarrow$ The characteristic $\mathrm{O}(2)$-bundle $E = L/I$.

$\mathrm{Pin}^{-}(2)$-monopole on X = I-invariant Seiberg-Witten on $\tilde{X}$

In fact, $\mathcal{M}_{\mathrm{Pin}} = (\mathcal{M}_{SW})^{I}$

$\mathrm{Pin}^-(2)$-monopole equations

For $\mathrm{O}(2)$-connection A on E and $\Phi \in \Gamma(S^+)$,

$$\begin{cases} D_A \Phi = 0, \\ F_A^+ = q(\Phi). \end{cases}$$

Gauge group for $\mathrm{Pin}^-(2)$-monopole

$$\mathcal{G} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1))$$

where $\{\pm 1\}$ acts on $\mathrm{U}(1)$ by complex conjugation.

$\mathrm{Pin}^-(2)$-monopole moduli space

$$\mathcal{M} = \{\text{solutions}\}/\mathcal{G}$$

Remark

- If $\tilde{X} \to X$ is a trivial covering
  $\Rightarrow P \xrightarrow{\mathrm{Spin}^{c-}(4)} X$ of a Spin^{c-}-str. c is reduced to a $\mathrm{Spin}^c(4)$-bundle
  $\to$ Spin^c-structure c' on X

$$\Rightarrow \boxed{\mathrm{Pin}^-(2)\text{-monopole on } c = \text{SW on } c'}$$

- Conversely, if it is given a Spin^c-structure

$$(P_c \xrightarrow{\mathrm{Spin}^c(4)} X, P_c/\,\mathrm{U}(1) \cong Fr(X))$$

$\Rightarrow P_c \times_{\mathrm{Spin}^c(4)} \mathrm{Spin}^{c-}(4)$ gives a Spin^{c-}-str. on the trivial double covering $\tilde{X} \to X$.

- The $\mathcal{G}$-action on (A, Φ) with $\Phi \not\equiv 0$ is free. $\rightarrow$ irreducible
- $\tilde{X}$: nontrivial $\Rightarrow$ the stabilizer of $(A, \Phi \equiv 0)$ is $\{\pm 1\}$.

  $\rightarrow$ $\{\pm 1\}$-reducible

- $\tilde{X}$: trivial $\Rightarrow$ the stabilizer of $(A, \Phi \equiv 0)$ is S^1. $\rightarrow$ S^1-reducible

- Let $l := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$.
  If $b_+^l = \dim H^+(X; l) \geq 1$ $\Rightarrow$ by perturbing the eqns
  - $\mathcal{M}$ contains no reducible,
  - $\mathcal{M}$ is a finite dimensional compact manifold.
  - $\tilde{X}$: nontrivial $\Rightarrow$ $\mathcal{M}$ may be nonorientable.

$\tilde{X}$: nontrivial $\Rightarrow$

$$\mathcal{M} \subset ((\Gamma(S^+) \setminus 0) \times \{\text{connections on } E\})/\mathcal{G} \underset{h.e.}{\simeq} \mathbb{RP}^\infty \times T^{b_1^l}$$

$\tilde{X}$: trivial $\Rightarrow$

$$\mathcal{M} \subset ((\Gamma(S^+) \setminus 0) \times \{\text{connections on } E\})/\mathcal{G} \underset{h.e.}{\simeq} \mathbb{CP}^\infty \times T^{b_1}$$

Suppose $\tilde{X}$: nontrivial

$H^*(\mathbb{RP}^\infty \times T^{b_1^l}; \mathbb{Z}_2) = \mathbb{Z}_2[\eta] \otimes \bigwedge \mathbb{Z}_2^{b_1^l}$, where η: generator of $H^1(\mathbb{RP}^\infty)$.

$\mathrm{Pin}^-(2)$-monopole invariants

Define $\mathrm{SW}^{\mathrm{Pin}} \colon \mathbb{Z}_2[\eta] \otimes \bigwedge \mathbb{Z}_2^{b_1^l} \to \mathbb{Z}_2$ by

$$\mathrm{SW}^{\mathrm{Pin}}(\eta^k \otimes t) = \langle \eta^k \otimes t, [\mathcal{M}] \rangle.$$

$\tilde{X}$: trivial $\Rightarrow$ May assume $\mathrm{SW}^{\mathrm{Pin}} = \mathrm{SW}$.

Theorem 1

- X: 4-manifold, for a Spin^c-str. c_1', SW-inv is odd.
  Let c_1 be the Spin^{c-}-str. associated to c_1'
- Y: 4-manifold with double covering $\tilde{Y}$, s.t.
  - $\exists$ positive scalar curvature metric
  - $\exists$ Spin^{c-}-str. c_2
    s.t. $b_+^l = 0$ & v-dim $\mathcal{M} = b_1^l$, for $l = \tilde{Y} \times_{\{\pm 1\}} \mathbb{Z}$.
    $\Rightarrow \mathcal{M} = \{$reducibles only$\}/\mathcal{G} \cong T^{b_1^l}$ & transversal

For $c_1 \# c_2$ over $X \# Y$,

$$\mathrm{SW}^{\mathrm{Pin}}(\eta \otimes t^{\mathsf{top}}) \neq 0,$$

where t^{top} is the generator of $H^{b_1^l}(T^{b_1^l})$.

- An example of $Y = (S^2 \times \Sigma_{g_1}) \# \cdots \# (S^2 \times \Sigma_{g_k})$.
- Main theorem is a corollary of Theorem 1.

Theorem 1

- X: 4-manifold, for a Spin^c-str. c_1', SW-inv is odd.
  Let c_1 be the Spin^{c-}-str. associated to c_1'
- Y: 4-manifold with double covering $\tilde{Y}$, s.t.
  - $\exists$ positive scalar curvature metric
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For $c_1 \# c_2$ over $X \# Y$,

$$\mathrm{SW}^{\mathrm{Pin}}(\eta \otimes t^{\text{top}}) \neq 0,$$

where t^{top} is the generator of $H^{b_1^l}(T^{b_1^l})$.

- The virtual dimension of $\mathcal{M}(X \# Y, c_1 \# c_2)$ is positive.
- The ordinary SW & stable cohomotopy SW of $X \# Y$ are 0.
  ($\because$ Y admits a PSC metric.)

Another application

Embedded surfaces representing a class in $H_2(X;l)$

- $\tilde{X} \to X$: nontrivial double covering, $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$.

Let us consider a connected surface Σ s.t.

- $i\colon \Sigma \hookrightarrow X$: embedding
- (The orientation coefficient of Σ) $= i^* l$

$\to$ $\exists$Fundamental class $[\Sigma] \in H_2(\Sigma; i^* l)$.
Let $\alpha := i_*[\Sigma] \in H_2(X;l)$, where $i_*\colon H_2(\Sigma; i^*l) \to H_2(X;l)$.

Proposition

For $\forall \alpha \in H_2(X;l)$, there exists Σ as above.

Remark

- Σ may be orientable or nonorientable.

Theorem 2 (N. 2011)

- (X, l, Σ) as above. Suppose $b_+^l \geq 2$.
- Let $[\Sigma] \in H_2(X; l)$.
  Suppose $[\Sigma] \cdot [\Sigma] \geq 0$, & $[\Sigma]$ is not a torsion.
- c: Spin^{c-}-structure $\rightarrow$ The associated $\mathrm{O}(2)$-bundle E
- $\tilde{c}$: the Spin^c-structure on $\tilde{X}$ induced from c.

If one of the following is nonzero

- $\mathrm{SW}^{\mathrm{Pin}}$ or stable cohomotopy $\mathrm{SW}^{\mathrm{Pin}}$ of (X, c),
- SW or stable cohomotopy SW of $(\tilde{X}, \tilde{c})$,

then

$$-\chi(\Sigma) \geq [\Sigma] \cdot [\Sigma] + |\tilde{c}_1(E) \cdot [\Sigma]|,$$

where $\tilde{c}_1(E) \in H^2(X; l)$ is the Euler class of E defined in $H^2(X; l)$, called the *twisted 1st Chern class*,

Example

- $X = K3\#(S^2 \times \Sigma_1)\# \cdots \#(S^2 \times \Sigma_k)$, $(g_i \geq 1)$.
- $\exists c$ s.t. $\tilde{c}_1(E) = 0$ & $\mathrm{SW}^{\mathrm{Pin}}(X, c) \neq 0$.

For $\Sigma \hookrightarrow X$ s.t. $[\Sigma] \in H_2(X; l)$ & $[\Sigma] \cdot [\Sigma] \geq 0$ & $[\Sigma]$ is not a torsion,

$$-\chi(\Sigma) \geq [\Sigma] \cdot [\Sigma].$$

Proof of Theorem 1

Simplest case

- $X_1 = K3$. For canonical Spin^c str., $\mathrm{SW}_{K3} = \pm 1$.
  Assume $\mathcal{M}_{X_1} = \{$ Only one irreducible class $\}$
- $X_2 = S^2 \times T^2$, $l \to X_2$ nontrivial $\Rightarrow b_1^l = b_+^l = 0$
  $\Rightarrow \mathcal{M}_{X_2} = \{\exists^1 \{\pm 1\}$-reducible class $\}$

Claim $\mathrm{SW}^{\mathrm{Pin}}_{X_1 \# X_2}(\eta) \neq 0$, $\eta \in H^1(\mathbb{RP}^\infty)$: generator

To prove

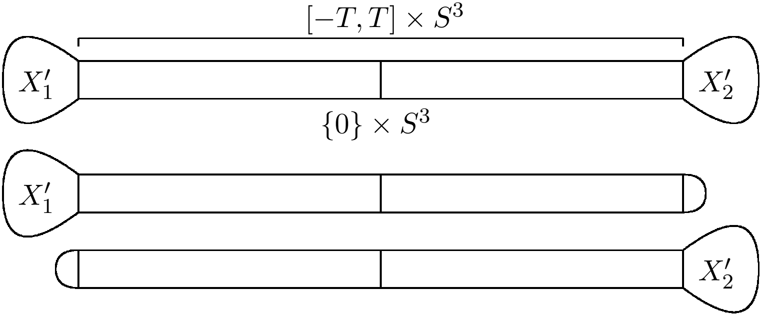
$$[\mathcal{M}_{X_1 \# X_2}] = [\mathbb{RP}^1] \in H_1(\mathbb{RP}^\infty)$$

Strategy = Gluing
Make solutions on $X_1 \# X_2$ by gluing solutions on X_1 and X_2.

- $X_1 \# X_2 = X_1' \cup_{S^3} X_2'$, where $X_i' = X_i \setminus D^4$.
- Insert a long cylinder $X_1' \cup ([-T,T] \times S^3) \cup X_2'$.
- Similarly, $X_1 = X_1' \cup ([-T,T] \times S^3) \cup D^4$,
  $X_2 = D^4 \cup ([-T,T] \times S^3) \cup X_2'$.

$\boxed{T \gg 1}$ For every solution (A, Φ) on X_1, X_2 or $X_1 \# X_2$,

$$(A,\Phi)|_{\{0\}\times S^3} \approx (\theta, 0): \ S^1\text{-reducible on } S^3$$



- Let (A_i, Φ_i) be a solution on X_i $(i = 1, 2)$.
- Cut off (A_i, Φ_i) near $\{0\} \times S^3$ to $(\theta, 0)$. $\longrightarrow (A_i', \Phi_i')$.

Glue (A_1', Φ_1') and (A_1', Φ_1') via a gluing parameter $\rho \in \Gamma = S^1$. $\Gamma =$ stabilizer of $(\theta, 0)$

$\Rightarrow$ approximate solution $(A_1', \Phi_1')\#_\rho(A_1', \Phi_1')$

$\Rightarrow$ Can find an exact solution (A_ρ, Φ_ρ) near the approx. solution.
(A_ρ, Φ_ρ) is unique.

Summary

(A_1, Φ_1) & (A_2, Φ_2) $\Rightarrow$ Γ-family of solutions on $X_1 \# X_2$

$$\{(A_\rho, \Phi_\rho)\}_{\rho \in \Gamma}$$

Proposition

$(A_\rho, \Phi_\rho) \sim (A_{\rho'}, \Phi_{\rho'}) \Leftrightarrow [\rho] = [\rho'] \in \Gamma/(\Gamma_1 \times \Gamma_2)$
where

- Γ_i = stabilizer of (A_i, Φ_i)
- $\Gamma_1 \times \Gamma_2 \curvearrowright \Gamma$: multiplication

- (A_1, Φ_1): irreducible $\Rightarrow \Gamma_1 = 1$
- $(A_2, \Phi_2 = 0)$: $\{\pm 1\}$-reducible $\Rightarrow \Gamma_2 = \{\pm 1\}$

$\Rightarrow \{\mathcal{G}$-equiv. classes of $(A_\rho, \Phi_\rho) \,|\, \rho \in \Gamma\} \cong S^1/\{\pm 1\}$

In fact,

$$\mathcal{M}_{X_1 \# X_2} \cong S^1/\{\pm 1\} \cong \mathbb{R}\mathrm{P}^1$$