$\mathrm{Pin}^-(2)$-monopole theory II

$\mathrm{Pin}^-(2)$-monopole invariants

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$$\mathrm{Pin}^-(2)\text{-monopole equations} \rightarrow \begin{cases} \mathrm{Pin}^-(2)\text{-monopole invariants} \\ \text{stable cohomotopy invariants} \end{cases}$$

$$\textit{Cf.}\ \text{Seiberg-Witten equations } (\mathrm{U}(1)\text{-monopole}) \rightarrow \begin{cases} \text{Seiberg-Witten invariants} \\ \text{stable cohomotopy invariants} \\ \text{[Bauer-Furuta]} \end{cases}$$

Applications

- Exotic structures
- Adjunction inequalities
- Calculation of the Yamabe invariants (j/w M. Ishida & S. Matsuo)

Exotic structures

- $E(1) = \mathbb{C}\mathrm{P}^2 \# 9\overline{\mathbb{C}\mathrm{P}}^2$
- $E(n) = \underbrace{E(1)\#_f \cdots \#_f E(1)}_{n}$ $\quad E(2) = K3$

Theorem 4 (N.'15)

For $\forall n \in \mathbb{Z}$, $n \geq 1$
$\exists \mathcal{S}_n$: a set of ∞'ly many distinct smooth structures on $E(n)$ s.t.
for $\forall Z := \underbrace{Z_1 \# \cdots \# Z_k}_{\text{arbitrary number}}$ where $Z_i = \begin{cases} S^2 \times \Sigma_g & (g \geq 1) \\ S^1 \times Y^3 & (b_1(Y) \geq 1) \end{cases}$
$\forall \sigma \neq \sigma' \in \mathcal{S}_n$, $\Rightarrow E(n)_\sigma \not\cong E(n)_{\sigma'}$

$$E(n)_\sigma \# Z \quad \underset{\text{not diffeo.}}{\not\cong} \quad E(n)_{\sigma'} \# Z$$

Remarks

- For $\forall\sigma$ & $\forall Z$, Donaldson & SW inv of $E(n)_\sigma \# Z = 0$
- If Z contains a component $Z_i = S^2 \times \Sigma_g$
  $\Rightarrow$ stable cohomotopy SW$= 0$
- But $\mathrm{Pin}^-(2)$-monopole inv of $E(n)_\sigma \# Z \neq 0$ for ∞'ly many σ

- [Wall] $\forall\sigma \neq \sigma'$, $\exists k$

$$E(n)_\sigma \# k(S^2 \times S^2) \underset{\text{diffeo.}}{\cong} E(n)_{\sigma'} \# k(S^2 \times S^2)$$

More remarks

Fact 1

X_1, X_2: closed oriented 4-manifolds with $b_+(X_1), b_+(X_2) \geq 1$
$\Rightarrow$ Donaldson & SW of $X_1 \# X_2 = 0$

Fact 2

$\exists W$: closed ori. 4-mfd with $b_+(W) = 0$ (e.g. $\overline{\mathbb{CP}}^2$, $\mathbb{Q}HS^4$, $S^1 \times S^3$)

$$\left\{ \begin{array}{c} \text{Donaldson} \\ \text{SW} \end{array} \right\} \text{of } X_1 \neq 0 \ \Rightarrow \left\{ \begin{array}{c} \text{Donaldson} \\ \text{SW} \end{array} \right\} \text{of } X_1 \# W \neq 0$$

- $\mathrm{Pin}^{-}(2)$-monopole = SW twisted along a local system ℓ.
- For Z in Theorem 1, $\exists \ell$ s.t. $b_{+}^{\ell}(Z) = 0 \leftarrow b_{+}$ in local coefficient

Theorem 5

$$\mathrm{SW}^{\mathrm{Pin}}(X_1 \# Z) \equiv \mathrm{SW}^{\mathrm{U}(1)}(X_1) \mod 2$$

where

- $\mathrm{SW}^{\mathrm{Pin}}(X_1 \# Z)(\in \mathbb{Z}/2)$: $\mathrm{Pin}^{-}(2)$-monopole invariant
- $\mathrm{SW}^{\mathrm{U}(1)}(X_1)(\in \mathbb{Z})$: ordinary SW invariant

Later we will state Theorem 5 more precisely

Calculation of Yamabe invariants

Theorem([LeBrun,'96,'99])

Let M be a compact minimal Kähler surface, $b_+ \geq 2$, $c_1^2(M) \geq 0$. Then

$$\mathcal{Y}(M) = \mathcal{Y}(M \# k\overline{\mathbb{C}\mathrm{P}^2}) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}.$$

- The proof uses the Seiberg-Witten equations.

- Note $c_1^2(M) = 2\chi(M) + 3\tau(M)$. χ: Euler, τ: signature

Theorem (Ishida-Matsuo-N.,'14)

Let M be a compact minimal Kähler surface, $b_+ \geq 2$, $c_1^2(M) \geq 0$.
Let $Z = Z_1 \# \cdots \# Z_k$ such that

$$Z_i = S^2 \times \Sigma \quad \text{or} \quad S^1 \times Y^3 \quad \text{or} \quad N$$

with $g(\Sigma) > 0$, $\mathcal{Y}(N) \geq 0$, $b_+(N) = 0$. (Ex. $N = \overline{\mathbb{C}\mathrm{P}^2}$)

Then $\qquad \mathcal{Y}(M \# Z) = \mathcal{Y}(M) = -4\sqrt{2}\pi\sqrt{c_1^2(M)}$.

- The proof uses $\mathrm{Pin}^-(2)$-monopole equations.
  Especially, $\mathrm{SW}^{\mathrm{Pin}^-(2)}(M \# Z) \underset{(2)}{\equiv} \mathrm{SW}^{\mathrm{U}(1)}(M) \neq 0$.

The adjunction inequality

Theorem

- X: closed ori. Spin^c 4-manifold with $b_+ \geq 2$.
- L: the determinant line bundle
- $\Sigma \subset X$: connected embedded surface s.t. $[\Sigma] \in H_2(X;\mathbb{Z})$, $[\Sigma]\cdot[\Sigma] \geq 0$.

If $\mathrm{SW}^{\mathrm{U}(1)}(X) \neq 0$ or stable cohomotopy $\mathrm{SW}^{\mathrm{U}(1)}(X) \neq 0$,

$$\Rightarrow \quad -\chi(\Sigma) = 2g - 2 \geq [\Sigma]\cdot[\Sigma] + |c_1(L)[\Sigma]|.$$

- This is due to: [Kronheimer-Mrowka], [Fintushel-Stern], [Morgan-Szabo-Taubes], [Ozsvath-Szabo], [Furuta-Kametani-Matsue-Minami]...

Embedded surfaces representing a class in $H_2(X;\ell)$

- $\tilde{X} \to X$: nontrivial double covering, $\ell = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$.

Consider a connected surface Σ s.t.

- $i\colon \Sigma \hookrightarrow X$: embedding
- (The orientation coefficient of Σ) $= i^*\ell$

$\to$ $\exists$Fundamental class $[\Sigma] \in H_2(\Sigma; i^*\ell)$.
Let $\alpha := i_*[\Sigma] \in H_2(X;\ell)$, where $i_*\colon H_2(\Sigma; i^*\ell) \to H_2(X;\ell)$.

Proposition

For $\forall \alpha \in H_2(X;\ell)$, there exists Σ as above.

Remark

- Σ may be orientable or nonorientable.

Theorem (N.)

- (X, ℓ, Σ) as above. Suppose $b_+^\ell = \dim H^+(X;\ell) \geq 2$.
- Let $[\Sigma] \in H_2(X;\ell)$.
  Suppose $[\Sigma]\cdot[\Sigma] \geq 0$, & $[\Sigma]$ is not a torsion.
- $\mathfrak{s}$: Spin^{c-}-structure on ℓ $\to$ The associated $\mathrm{O}(2)$-bundle E

$$\mathrm{SW}^{\mathrm{Pin}}(X,\mathfrak{s}) \neq 0 \;\Rightarrow\; -\chi(\Sigma) \geq [\Sigma]\cdot[\Sigma] + |\tilde{c}_1(E)\cdot[\Sigma]|$$

Recall:

- Seiberg-Witten equations are defined on a Spin^c-structure.

$$\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$$

- $\mathrm{Pin}^-(2)$-monopole eqns are defined on a Spin^{c-}-structure.

$$\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$$

Spin^{c-}-structures

- X: an oriented Riemannian 4-manifold.
  $\longrightarrow Fr(X)$: The $\mathrm{SO}(4)$-frame bundle.
- $\tilde{X} \overset{2:1}{\to} X$: double covering, $\ell := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$

[Furuta,08] A Spin^{c-}-structure $\mathfrak{s}$ on $\tilde{X} \to X$ is given by

- P: a $\mathrm{Spin}^{c-}(4)$-bundle over X,
- $P/\mathrm{Spin}^{c}(4) \overset{\cong}{\to} \tilde{X}$
- $P/\mathrm{Pin}^{-}(2) \overset{\cong}{\to} Fr(X)$.
- $E = P/\mathrm{Spin}(4) \overset{\mathrm{O}(2)}{\to} X$: characteristic $\mathrm{O}(2)$-bundle.
  $\longrightarrow \ell$-coefficient Euler class $\tilde{c}_1(E) \in H^2(X;\ell)$.

- Consider a trivial double cover $\tilde{X} = X \sqcup X \to X$.
- A Spin^{c-}-structure on $X \sqcup X \to X$ has a canonical reduction to a Spin^{c}-structure on X.
- We call such a Spin^{c-}-structure untwisted.
- Spin^{c-} on nontrivial $\tilde{X}$ is called twisted.
- Often identify

$$\text{an untwisted } \mathrm{Spin}^{c-} \; = \; \text{its canonical reduction } \mathrm{Spin}^{c}$$

Furthermore

$$\mathrm{Pin}^{-}(2)\text{-monopole on an untwisted } \mathrm{Spin}^{c-}$$
$$= \mathrm{U}(1)\text{-monopole on the canonical reduction } \mathrm{Spin}^{c}$$

(ordinary SW)

Theorem 5 (revisited)

X_1: closed oriented 4-manifold with a Spin^c structure $\mathfrak{s}_1$
Suppose $b_+(X_1) \geq 2$

$Z := Z_1 \# \cdots \# Z_k$ where $Z_i = \begin{cases} S^2 \times \Sigma_g & (g \geq 1) \\ S^1 \times Y^3 & (b_1(Y) \geq 1) \end{cases}$

$\exists\,\mathrm{Spin}^{c-}$ structure $\mathfrak{s}'$ on Z s.t.

$$\mathrm{SW}^{\mathrm{Pin}}(X_1 \# Z, \mathfrak{s}_1 \# \mathfrak{s}') \equiv \mathrm{SW}^{\mathrm{U}(1)}(X_1, \mathfrak{s}_1) \mod 2$$

where

- $\mathrm{SW}^{\mathrm{Pin}}(X_1 \# Z) (\in \mathbb{Z}/2)$: $\mathrm{Pin}^-(2)$-monopole invariant
- $\mathrm{SW}^{\mathrm{U}(1)}(X_1) (\in \mathbb{Z})$: ordinary SW invariant

Below we explain the proof of Theorem 5

Monopole map

- For simplicity, we assume $b_1^\ell = \dim H_1(X;\ell) = 0$.
- Fix a reference $\mathrm{O}(2)$-connection A on E

$$\begin{aligned}\mu\colon \Gamma(S^+)\times\Omega(\ell\otimes i\mathbb{R}) &\to \Gamma(S^-)\times(\Omega^+\oplus\Omega^0)(\ell\otimes i\mathbb{R})\\ (a,\phi) &\mapsto (D_{A+a}\phi, F^+_{A+a}-q(\phi), d^*a)\end{aligned}$$

- μ is $\{\pm 1\}$-equivariant for twisted Spin^{c-}
  $\mathrm{U}(1)$-equivariant for untwisted Spin^{c-}
- μ^{-1}(ball) $\subset$ ball
- The moduli space
  - $\mathcal{M} = \mu^{-1}(0)/\{\pm 1\}$ (may be nonorientable)
  - $\mathcal{M} = \mu^{-1}(0)/\,\mathrm{U}(1)$ (orientable, identified with $\mathrm{U}(1)$-monopole moduli)

Finite dimensional approximation

- Decompose $\mu = \mathcal{D} + \mathcal{Q}$ as $\mathcal{D}$: linear & $\mathcal{Q}$: quadratic
- Fix $\lambda \gg 1$.

$$V_\lambda = \mathrm{Span}\begin{pmatrix}\text{eigenspaces of } \mathcal{D}^*\mathcal{D} \\ \text{eigenvalues} < \lambda\end{pmatrix}$$

$$W_\lambda = \mathrm{Span}\begin{pmatrix}\text{eigenspaces of } \mathcal{D}\mathcal{D}^* \\ \text{eigenvalues} < \lambda\end{pmatrix}$$

- $p_\lambda \colon \Gamma(S^-) \times (\Omega^+ \oplus \Omega^0)(\ell \otimes i\mathbb{R}) \to W_\lambda$, L^2-projection
- Finite dim approx. $\boxed{f = \mathcal{D} + p_\lambda \mathcal{Q} \colon V_\lambda \to W_\lambda}$
- f is $\{\pm 1\}$(or $\mathrm{U}(1)$)-equivariant, proper

$$f\colon \tilde{\mathbb{R}}^{x+a}\oplus\mathbb{R}^{y}\to\tilde{\mathbb{R}}^{x}\oplus\mathbb{R}^{y+b},\quad a=\operatorname{ind}_{\mathbb{R}} D_A,\ b=b_+^{\ell}(X)$$

$$\{\pm 1\} \text{ acts on } \begin{cases} \tilde{\mathbb{R}} \text{ by multiplication} \\ \mathbb{R} \text{ trivially} \end{cases}$$

N.B. $\tilde{\mathbb{R}}^{\bullet}$: spinor part, $\mathbb{R}^{\bullet}$: form part

$$f\colon \mathbb{C}^{x+a}\oplus\mathbb{R}^{y}\to\mathbb{C}^{x}\oplus\mathbb{R}^{y+b},\quad a=\operatorname{ind}_{\mathbb{C}} D_A,\ b=b_+(X)$$

$$\mathrm{U}(1) \text{ acts on } \begin{cases} \mathbb{C} \text{ by multiplication} \\ \mathbb{R} \text{ trivially} \end{cases}$$

N.B. $\mathbb{C}^{\bullet}$: spinor part, $\mathbb{R}^{\bullet}$: form part

[Fact] $f|_{\{0\}\oplus\mathbb{R}^y}$ is a linear inclusion

- The "moduli spaces"

$$\left.\begin{aligned} \frac{f^{-1}(0)}{\{\pm 1\}} &\subset \frac{\tilde{\mathbb{R}}^{x+a} \oplus \mathbb{R}}{\{\pm 1\}} \\ \frac{f^{-1}(0)}{\mathrm{U}(1)} &\subset \frac{\mathbb{C}^{x+a} \oplus \mathbb{R}}{\mathrm{U}(1)} \end{aligned}\right\} \ni (0, c) \text{ is a quotient singularity}$$

(reducible)

- Suppose $b \geq 1 \Rightarrow$ perturb f by $v \in \operatorname{Im}\left(f|_{\{0\} \oplus \mathbb{R}^y}\right)^{\perp} \cong \mathbb{R}^b$
  $\Rightarrow (f + v)^{-1}(0)$ contains no reducible

- Assume transversality

$$M(X) = \frac{(f+v)^{-1}(0)}{\{\pm 1\}} \subset \frac{(\tilde{\mathbb{R}}^{x+a} \setminus \{0\}) \oplus \mathbb{R}}{\{\pm 1\}} \underset{h.e.}{\simeq} \mathbb{R}\mathrm{P}^{\bullet}$$

$$M(X) = \frac{(f+v)^{-1}(0)}{\mathrm{U}(1)} \subset \frac{(\mathbb{C}^{x+a} \setminus \{0\}) \oplus \mathbb{R}}{\mathrm{U}(1)} \underset{h.e.}{\simeq} \mathbb{C}\mathrm{P}^{\bullet}$$

- $M(X)$ is a compact manifold
- Define

$$\mathrm{SW}^{\mathrm{Pin}}(X) = \langle \alpha, [M(X)]_2 \rangle \in \mathbb{Z}/2, \quad \alpha \in H^*(\mathbb{R}\mathrm{P}^{\bullet}; \mathbb{Z}/2)$$

$$\mathrm{SW}^{\mathrm{U}(1)}(X) = \langle \alpha, [M(X)] \rangle \in \mathbb{Z}, \quad \alpha \in H^*(\mathbb{C}\mathrm{P}^{\bullet}; \mathbb{Z})$$

[Remark] if $b \geq 2 \Rightarrow \mathrm{SW}^{\mathrm{Pin}}$ & $\mathrm{SW}^{\mathrm{U}(1)}$ are diffeomorphism inv.

Connected sum $X_1 \# X_2$

Untwisted cases (oridinary SW)

- $f_{X_1\#X_2}$, f_{X_1}, f_{X_2}: finite dim approx for $X_1\#X_2$, X_1, X_2

[Bauer]

$$f_{X_1\#X_2} \underset{\mathrm{U}(1)\text{-h.e.}}{\simeq} f_{X_1} \times f_{X_2}$$

Case 1. $b_+(X_1)$, $b_+(X_2) \geq 1$ & $\dim \frac{f_{X_1}^{-1}(0)}{\mathrm{U}(1)} = \dim \frac{f_{X_1}^{-1}(0)}{\mathrm{U}(1)} = 0$

$$\Rightarrow \dim M(X_1\#X_2) = \dim \frac{(f_{X_1} \times f_{X_2})^{-1}(0)}{\mathrm{U}(1)} = 1$$

$$H_1(\mathbb{C}\mathrm{P}^\bullet) = 0 \qquad \therefore \mathrm{SW}^{\mathrm{U}(1)}(X_1\#X_2) = 0$$

Case 2. $X_2 = \overline{\mathbb{C}\mathrm{P}}^2$ $(b_1 = 0,\ b_+ = 0)$ & $\dim \frac{f_{X_2}^{-1}(0)}{\mathrm{U}(1)} = -1$

$\Rightarrow f_{X_2}^{-1}(0) = \{0\} \leftarrow$ only one reducible

$$\frac{f_{X_1 \# X_2}^{-1}(0)}{\mathrm{U}(1)} = \frac{f_{X_1}^{-1}(0) \times \{0\}}{\mathrm{U}(1)} = \frac{f_{X_1}^{-1}(0)}{\mathrm{U}(1)}$$

$$\therefore \mathrm{SW}^{\mathrm{U}(1)}(X_1 \# \overline{\mathbb{C}\mathrm{P}}^2) = \mathrm{SW}^{\mathrm{U}(1)}(X_1)$$

(Twisted # untwisted) case

$$\tilde{X}_1 \to X_1: \text{ nontrivial}$$
$$\tilde{X}_2 = X_2 \sqcup X_2 \to X_2: \text{ trivial}$$

e.g. $X_1 \# X_2 = (S^2 \times T^2) \# E(n)$

- f_{X_1}, $f_{X_1 \# X_2}$: $\{\pm 1\}$-equivariant
- f_{X_2}: $\mathrm{U}(1)$-equivariant $\to$ also $\{\pm 1\}$-equivariant
  $(\because \{\pm 1\} \subset \mathrm{U}(1))$

Proposition

$$f_{X_1 \# X_2} \underset{\{\pm 1\}\text{-h.e.}}{\simeq} f_{X_1} \times f_{X_2}$$

Suppose $b^{\ell}_{+}(X_1) = 0$ & $\dim \frac{f_{X_1}^{-1}(0)}{\{\pm 1\}} = 0$

$$\Rightarrow f_{X_1}^{-1}(0) = \underset{\text{reducible}}{\{0\}} \cup \underbrace{\{\pm v_1\} \cup \cdots \cup \{\pm v_k\}}_{\text{irreducibles}} \quad (v_i \neq 0)$$

Suppose $b_{+}(X_2) \geq 1$ & $\dim \frac{f_{X_2}^{-1}(0)}{\mathrm{U}(1)} = 0 \Rightarrow f_{X_2}^{-1}(0) = \underbrace{S^1 \cup \cdots \cup S^1}_{\text{irreducibles}}$

$$\frac{\left(\tilde{\mathbb{R}}^{\bullet} \times \mathbb{C}^{\bullet}\right) \setminus \{0\}}{\{\pm 1\}} \supset \frac{\{0\} \times S^1}{\{\pm 1\}} \to \text{ nonzero in } H_1(\mathbb{R}\mathrm{P}^{\bullet})$$

$$\supset \frac{\{\pm v\} \times S^1}{\{\pm 1\}} \to \text{ null-homologous}$$

$$\Rightarrow \mathrm{SW}^{\mathrm{Pin}}(X_1 \# X_2) \underset{(2)}{\equiv} \mathrm{SW}^{\mathrm{U}(1)}(X_2)$$

Stable cohomotopy $\mathrm{Pin}^-(2)$-monopole invariants

$$f\colon V=\tilde{\mathbb{R}}^{x+a}\oplus\mathbb{R}^{y}\to W=\tilde{\mathbb{R}}^{x}\oplus\mathbb{R}^{y+b}\quad \{\pm1\}\text{-equiv., proper}$$

$$\widetilde{\mathrm{SW}}^{\mathrm{Pin}}(X):=[f^+]\in\left\{S^V,S^W\right\}^{\{\pm1\}}\underset{\text{if } a\geq 1}{\cong}\left\{S^V/\{\pm1\},S^W\right\}$$
$$\cong\left[\mathbb{R}\mathrm{P}^{a-1},S^{b-1}\right]=\pi^{a-1-d}(\mathbb{R}\mathrm{P}^{a-1})$$

where $d=a-b=$ the dim of the moduli sp.

$$\boxed{d=0}\quad \pi^{a-1-d}(\mathbb{R}\mathrm{P}^{a-1})\cong\begin{cases}\mathbb{Z} & a:\ \text{even}\\ \mathbb{Z}_2 & a:\ \text{odd}\end{cases}\leftrightarrow H^{a-1}(\mathbb{R}\mathrm{P}^{a-1})$$

$$[f]\mapsto\deg\left(f/\{\pm1\}\right)=\#\left(\frac{f^{-1}(0)}{\{\pm1\}}\right)=\mathrm{SW}^{\mathrm{Pin}}(X)$$

$d = 1$, a: even $\qquad \pi^{a-1-d}(\mathbb{RP}^{a-1}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

- By Atiyah-Hirzebruch spectral sequence, ∃surjective hom

$$\varphi\colon \pi^{a-1-d}(\mathbb{RP}^{a-1}) \twoheadrightarrow H^{a-2}(\mathbb{RP}^{a-1}) \cong H_1(\mathbb{RP}^{a-1}) \cong \mathbb{Z}_2$$
$$[f] \mapsto \mathrm{SW}^{\mathrm{Pin}}(X)$$

- $\ker\varphi \cong \mathbb{Z}_2$.

Theorem 3 (N.'16)

K: $K3$, $\quad E$: Enriques surface
For $X = K\#E\#k\overline{\mathbb{CP}}^2$ $(a = b + 1 = 6)$,

$$\widetilde{\mathrm{SW}}^{\mathrm{Pin}}(X) = [f_X] \neq 0, \text{ but } \varphi([f_X]) = \mathrm{SW}^{\mathrm{Pin}}(X) = 0.$$

Corollary

For $X = K\#E\#k\overline{\mathbb{C}\mathrm{P}}^2$,

- $\exists$exotic structures on X,
- the adjunction inequality holds,
- the Yamabe invariant of X is 0.