$\mathrm{Pin}^{-}(2)$-monopole theory I

Intersection forms with local coefficients

Nobuhiro Nakamura

Osaka Medical College

Dec 12, 2016

- $\mathrm{Pin}^-(2) = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}$
- $\mathrm{Pin}^-(2)$-monopole equations are a twisted version of the Seiberg-Witten ($\mathrm{U}(1)$-monopole) equations.

Applications of SW equations

- Intersection forms
  - Diagonalization theorem
  - The $10/8$ inequality for spin [Furuta]
- SW invariants
  - Exotic structures
  - Adjunction inequalities [Kronheimer-Mrowka] et al.
  - SW=GW [Taubes]
  - Calculation of the Yamabe invariants [LeBrun] et al
- stable cohomotopy invariants [Bauer-Furuta]

- $\mathrm{Pin}^-(2) = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}$
- $\mathrm{Pin}^-(2)$-monopole equations are a twisted version of the Seiberg-Witten ($\mathrm{U}(1)$-monopole) equations.

Applications of $\mathrm{Pin}^-(2)$-monopole equations

- Intersection forms with local coefficients
  - Diagonalization theorem in local coefficients
  - $10/8$-type inequality (for non-spin)
- $\mathrm{Pin}^-(2)$-monopole invariants
  - Exotic structures
  - Adjunction inequalities
  - Calculation of the Yamabe invariants (j/w M. Ishida & S. Matsuo)
- stable cohomotopy invariants

Today

- Intersection forms with local coefficients
  - Diagonalization theorem in local coefficients
  - $10/8$-type inequality (for non-spin)
  - $\mathbb{Z}_2$-Froyshov invariants & 4-manifolds with boundary

Next talk

- $\mathrm{Pin}^{-}(2)$-monopole invariants
- stable cohomotopy invariants

Intersection forms with local coefficients

- X: closed oriented 4-manifold with double cover $\tilde{X} \to X$
- $\ell := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, a $\mathbb{Z}$-bundle over X.
  $\longrightarrow H^*(X;\ell)$: ℓ-coefficient cohomology.
- Note $\ell \otimes \ell = \mathbb{Z}$. The cup product

  $$\cup\colon H^2(X;\ell) \times H^2(X;\ell) \to H^4(X;\mathbb{Z}) \cong \mathbb{Z},$$

  induces the intersection form with local coefficient

  $$Q_{X,\ell}\colon H^2(X;\ell)/\text{torsion} \times H^2(X;\ell)/\text{torsion} \to \mathbb{Z}.$$

- $Q_{X,\ell}$ is a symmetric bilinear unimodular form.

Theorem [Froyshov,2012]

- X: a closed connected oriented smooth 4-manifold s.t.

$$\begin{cases} H^2(X;\ell) \text{ contains no element of order } 4 \\ b^+(X)+\dim_{\mathbb{Z}/2}(\mathrm{tor} H_1(X;\mathbb{Z})\otimes\mathbb{Z}/2) \le 2. \end{cases} \quad (1)$$

- $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

$$\boxed{\text{If } Q_{X,\ell} \text{ is definite} \Rightarrow Q_{X,\ell} \sim \text{diagonal.}}$$

- The original form of Froyshov's theorem is:

$$\text{If } X \text{ with } \partial X = Y : \mathbb{Z}HS^3 \text{ satisfies (1)}$$
$$\& \ Q_{X,\ell} \text{ is nonstandard definite}$$
$$\Rightarrow \delta_0 \colon HF^4(Y;\mathbb{Z}/2) \to \mathbb{Z}/2 \text{ is non-zero.}$$

- $Y = S^3 \Rightarrow HF^4(Y;\mathbb{Z}/2) = 0 \Rightarrow$The above result.

- The proof uses $\mathrm{SO}(3)$-Yang-Mills theory on a $\mathrm{SO}(3)$-bundle V.
- Twisted reducibles (stabilizer $\cong \{\pm 1\}$) play an important role.
  V is reduced to $\lambda \oplus E$, where E is an $\mathrm{O}(2)$-bundle,
  $\lambda = \det E$: a nontrivial $\mathbb{R}$-bundle.

Cf [Fintushel-Stern'84]'s proof of Donaldson's theorem also used $\mathrm{SO}(3)$-Yang-Mills.
$\longrightarrow$ Abelian reducibles (stabilizer $\cong \mathrm{U}(1)$)
V is reduced to $\underline{\mathbb{R}} \oplus L$, where L is a $\mathrm{U}(1)$-bundle.

Theorem 1.(N. 2013)

- X: a closed connected ori. smooth 4-manifold.
- $\ell \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\ell \otimes \mathbb{R})^2 = 0$.

$$\boxed{\text{If } Q_{X,\ell} \text{ is definite} \Rightarrow Q_{X,\ell} \sim \text{diagonal.}}$$

Cf. Froyshov's theorem

- X: — s.t. $\begin{cases} H^2(X;\ell) \text{ contains no element of order } 4 \\ b^+(X) + \dim_{\mathbb{Z}/2}(\mathrm{tor} H_1(X;\mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2. \end{cases}$
- $\ell \to X$: a nontrivial $\mathbb{Z}$-bundle.

$$\boxed{\text{If } Q_{X,\ell} \text{ is definite} \Rightarrow Q_{X,\ell} \sim \text{diagonal.}}$$

Theorem 1.(N. 2013)

- X: a closed connected ori. smooth 4-manifold.
- $\ell \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\ell \otimes \mathbb{R})^2 = 0$.

If $Q_{X,\ell}$ is definite $\Rightarrow Q_{X,\ell} \sim$ diagonal.

- The proof uses $\mathrm{Pin}^-(2)$-monopole equations.
- $\mathrm{Pin}^-(2)$-monopole eqns are defined on a Spin^{c-} structure.
- Spin^{c-} structure is a $\mathrm{Pin}^-(2)$-variant of Spin^c-structure whose complex structure is "twisted along ℓ".

- The moduli space of $\mathrm{Pin}^-(2)$-monopoles is compact.
  $\longrightarrow$ Bauer-Furuta theory can be developed.

Furuta's theorem

Let X be a closed ori. smooth spin 4-manifold with indefinite Q_X.

$$b_2(X) \geq \frac{10}{8}|\operatorname{sign}(X)| + 2.$$

Theorem 2(N.'13, Furuta)

Let X be a closed connected ori. smooth 4-manifold. For any nontrivial $\mathbb{Z}$-bundle $\ell \to X$ s.t. $w_1(\ell \otimes \mathbb{R})^2 = w_2(X)$.

$$b_2^\ell(X) \geq \frac{10}{8}|\operatorname{sign}(X)| + 2,$$

where $b_2^\ell(X) = \operatorname{rank} H_2(X;\ell)$.

$\mathrm{Pin}^-(2)$-monopole equations

- Seiberg-Witten equations are defined on a Spin^c-structure.

$$\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$$

- $\mathrm{Pin}^-(2)$-monopole eqns are defined on a Spin^{c-}-structure.

$$\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$$

$\mathrm{Spin}^{c-}(4)$

$$\mathrm{Pin}^-(2) = \langle \mathrm{U}(1), j \rangle = \mathrm{U}(1) \cup j\,\mathrm{U}(1) \subset \mathrm{Sp}(1) \subset \mathbb{H}.$$

Two-to-one homomorphism $\mathrm{Pin}^-(2) \to \mathrm{O}(2)$

$$z \in \mathrm{U}(1) \subset \mathrm{Pin}^-(2) \mapsto z^2 \in \mathrm{U}(1) \cong \mathrm{SO}(2) \subset \mathrm{O}(2),$$
$$j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition $\mathrm{Spin}^{c-}(4) := \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$.

- $\mathrm{Spin}^{c-}(4)/\mathrm{Pin}^-(2) = \mathrm{Spin}(4)/\{\pm 1\} = \mathrm{SO}(4)$
- $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}(4) = \mathrm{O}(2)$
- The id. compo. of $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1) = \mathrm{Spin}^c(4)$

  $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}^c(4) = \{\pm 1\}$.

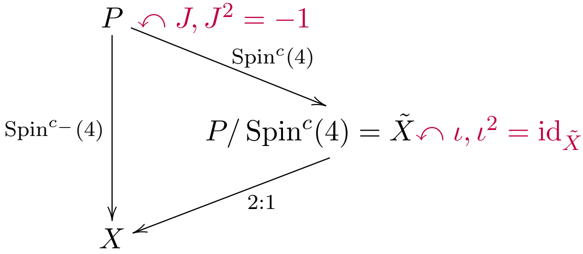
Spin^{c-}-structures

- X: an oriented Riemannian 4-manifold.
  $\longrightarrow Fr(X)$: The $\mathrm{SO}(4)$-frame bundle.
- $\tilde{X} \overset{2:1}{\to} X$: (nontrivial) double covering, $\ell := \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$

[Furuta,08] A Spin^{c-}-structure $\mathfrak{s}$ on $\tilde{X} \to X$ is given by

- P: a $\mathrm{Spin}^{c-}(4)$-bundle over X,
- $P/\mathrm{Spin}^c(4) \overset{\cong}{\to} \tilde{X}$
- $P/\mathrm{Pin}^-(2) \overset{\cong}{\to} Fr(X)$.

- $E = P/\mathrm{Spin}(4) \overset{\mathrm{O}(2)}{\to} X$: characteristic $\mathrm{O}(2)$-bundle.
  $\longrightarrow$ ℓ-coefficient Euler class $\tilde{c}_1(E) \in H^2(X;\ell)$.



- $P \xrightarrow{\mathrm{Spin}^c(4)} \tilde{X}$ defines a Spin^c-structure $\tilde{\mathfrak{s}}$ on $\tilde{X}$
- $J = [1, j] \in \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2) = \mathrm{Spin}^{c-}(4) \Rightarrow J$ covers ι
- Involution I on the spinor bundles $\tilde{S}^{\pm}$ of $\tilde{\mathfrak{s}}$:

$$\tilde{S}^{\pm} = P \times_{\mathrm{Spin}^c(4)} \mathbb{H}_{\pm} \curvearrowleft [J, j] =: I$$

$\Rightarrow$ $I^2 = 1$ & I is antilinear.

$\Rightarrow$ $S^{\pm} = \tilde{S}^{\pm}/I$ are the spinor bundles for the Spin^{c-}-str. $\mathfrak{s}$
$S^{\pm}$ are not complex bundles.

- Twisted Clifford multiplication

$$\rho\colon T^*X \otimes (\ell \otimes \sqrt{-1}\mathbb{R}) \to \mathrm{End}(S^+ \oplus S^-)$$

- An $\mathrm{O}(2)$-connection A on E + Levi-Civita $\Rightarrow$ Dirac operator

$$D_A\colon \Gamma(S^+) \to \Gamma(S^-)$$

- Weitzenböck formula

$$D_A^2\Phi = \nabla_A^*\nabla_A\Phi + \frac{s_g}{4}\Phi + \frac{\rho(F_A^+)}{2}\Phi$$

$\mathrm{Pin}^{-}(2)$-monopole equations

$$\begin{cases} D_A\Phi = 0, \\ \rho(F_A^+) = q(\Phi), \end{cases}$$

where

- A: $\mathrm{O}(2)$-connection on E & $\Phi \in \Gamma(S^+)$
- $F_A^+ \in \Omega^+(\ell \otimes \sqrt{-1}\mathbb{R})$
- $q(\Phi) = $ "$(\Phi^* \otimes \Phi) - \frac{1}{2}|\Phi|^2 \,\mathrm{id}$" $\in \mathrm{End}(S^+)$

Remark

$\mathrm{Pin}^{-}(2)$-monopole on X = I-invariant Seiberg-Witten on $\tilde{X}$

Gauge symmetry

$$\begin{aligned}\mathcal{G}_{\mathrm{Pin}} =&\Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1))\\ =&\{f \in \mathrm{Map}(\tilde{X}, \mathrm{U}(1)) \mid f(\iota x) = f(x)^{-1}\}\end{aligned}$$

where $\{\pm 1\} \curvearrowright \mathrm{U}(1)$ by $z \mapsto z^{-1}$.

Cf. Ordinary SW ($\mathrm{U}(1)$) case

$$\mathcal{G}_{\mathrm{U}(1)} = \mathrm{Map}(X, \mathrm{U}(1))$$

Moduli spaces

$\mathcal{M}_{\mathrm{Pin}} = \{$ solutions $\}/\mathcal{G}_{\mathrm{Pin}} \subset (\mathcal{A}_E \times \Gamma(S^+))/\mathcal{G}_{\mathrm{Pin}}$
where $\mathcal{A}_E$ = the space of $\mathrm{O}(2)$-connections on E

Proposition

- $\mathcal{M}_{\mathrm{Pin}}$ is compact.
- The virtual dimension of $\mathcal{M}_{\mathrm{Pin}}$:

$$d = \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b_0^l - b_1^l + b_+^l).$$

where $b_\bullet^l = \mathrm{rank}\, H_\bullet(X; l)$.

- If l is nontrivial & X connected $\Rightarrow b_0^l = 0$.

Reducibles

- Irreducible: (A, Φ), $\Phi \not\equiv 0 \leftarrow \mathcal{G}_{\mathrm{Pin}}$-action is free.
- reducible: $(A, \Phi \equiv 0) \leftarrow$ The stabilizer is $\{\pm 1\}$
- $\{$ reducible solutions $\}/\mathcal{G}_{\mathrm{Pin}} \cong T^{b_1^l} \subset \mathcal{M}_{\mathrm{Pin}}$.

Key difference

Ordinary SW($\mathrm{U}(1)$) case

- Reducible $\rightarrow$ The stabilizer $= \mathrm{U}(1)$.

$$\mathcal{M}_{\mathrm{U}(1)} \setminus \{\text{reducibles}\} \subset (\mathcal{A}_{\mathrm{U}(1)} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G}_{\mathrm{U}(1)} \simeq B\mathcal{G}_{\mathrm{U}(1)} \simeq T^{b_1} \times \mathbb{C}\mathrm{P}^\infty .$$

$\mathrm{Pin}^-(2)$-monopole case

- Reducible $\rightarrow$ The stabilizer $= \{\pm 1\}$.

$$\mathcal{M}_{\mathrm{Pin}} \setminus \{\text{reducibles}\} \subset (\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G}_{\mathrm{Pin}} \simeq B\mathcal{G}_{\mathrm{Pin}} \simeq T^{b_1^l} \times \mathbb{R}\mathrm{P}^\infty .$$

Proof of Theorem 1

Recall

Theorem 1.

- X: a closed connected ori. smooth 4-manifold.
- $\ell \to X$: a nontrivial $\mathbb{Z}$-bundle s.t. $w_1(\ell \otimes \mathbb{R})^2 = 0$.

If $Q_{X,\ell}$ is definite $\Rightarrow Q_{X,\ell} \sim$ diagonal.

- For simplicity, assume $b_1^l = 0$.

Lemma 1

$\forall$ characteristic elements w of $Q_{X,l}$,

$$0 \geq -|w^2| + b_2^l.$$

An element w in a unimodular lattice L is called *characteristic* if $w \cdot v \equiv v \cdot v \bmod 2$ for $\forall v \in L$.

Lemma 1 & [Elkies '95] $\Rightarrow Q_{X,l} \sim$ diagonal.

Lemma 2

If $w_1(l \otimes \mathbb{R})^2 = 0$
$\Rightarrow \forall$ characteristic element w, $\exists\, \mathrm{Spin}^{c-}$-str. s.t. $\tilde{c}_1(E) = w$.

Lemma 3

If $b^l_+ = b^l_1 = 0 \Rightarrow \dim \mathcal{M}_{\mathrm{Pin}} \leq 0$ for $\forall\, \mathrm{Spin}^{c-}$-str.

Lemma 2 & 3 $\Rightarrow$ Lemma 1

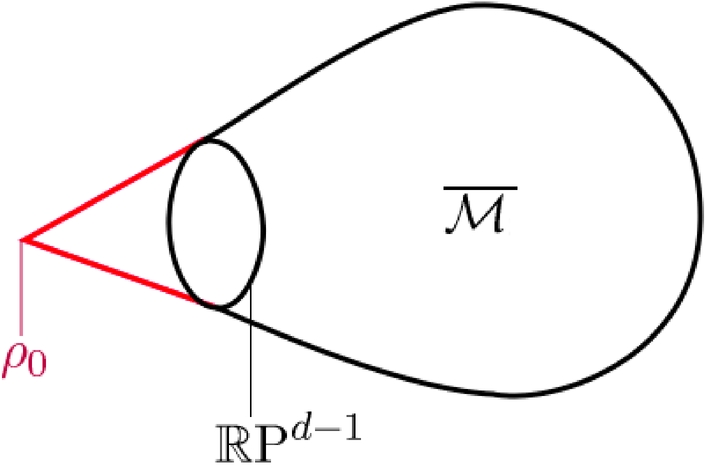
$$\begin{aligned} 0 \geq \dim \mathcal{M}_{\mathrm{Pin}} =& \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b^l_0 - b^l_1 + b^l_+) \\ =& \frac{1}{4}(-|w^2| + b^l_2) \end{aligned}$$

The structure of $\mathcal{M}_{\mathrm{Pin}}$ when $b_+(X;l) = 0$

- $b_1(X,l) = 0 \Rightarrow \exists^1$ reducible class $\rho_0 \in \mathcal{M}_{\mathrm{Pin}}$.
- Perturb the $\mathrm{Pin}^-(2)$-monopole equations by adding $\eta \in \Omega^+(i\lambda)$ to the curvature equation. $\rightarrow F_A^+ = q(\phi) + \eta$.
- For generic η, $\mathcal{M}_{\mathrm{Pin}} \setminus \{\rho_0\}$ is a d-dimensional manifold.
- Fix a small neighborhood $N(\rho_0)$ of $\{\rho_0\}$.

  $\Rightarrow N(\rho_0) \cong \mathbb{R}^d/\{\pm 1\} =$ a cone of $\mathbb{RP}^{d-1}$

  Then $\overline{\mathcal{M}}_{\mathrm{Pin}} := \overline{\mathcal{M}_{\mathrm{Pin}} \setminus N(\rho_0)}$ is a compact d-manifold & $\partial\overline{\mathcal{M}}_{\mathrm{Pin}} = \mathbb{RP}^{d-1}$.


$\overline{\mathcal{M}}$
ρ_0
$\mathbb{RP}^{d-1}$

- Note $\overline{\mathcal{M}}_{\mathrm{Pin}} \subset \left(\mathcal{A} \times \left(\Gamma(S^+) \setminus \{0\}\right)\right)/\mathcal{G} =: \mathcal{B}^*$.
- Recall $\mathcal{B}^* \underset{h.e.}{\simeq} T^{b_1(X;l)} \times \mathbb{R}\mathrm{P}^\infty$.

Lemma 3

If $b_+^l = 0$ & $b_1^l = 0 \Rightarrow d = \dim \mathcal{M}_{\mathrm{Pin}} \leq 0$.

Proof

- Suppose $d > 0$.
- Recall $\overline{\mathcal{M}}_{\mathrm{Pin}}$ is a compact d-manifold s.t. $\partial \overline{\mathcal{M}}_{\mathrm{Pin}} = \mathbb{R}\mathrm{P}^{d-1}$.
- $\exists C \in H^{d-1}(\mathcal{B}^*; \mathbb{Z}/2) \cong H^{d-1}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$ s.t. $\langle C, [\partial \overline{\mathcal{M}}_{\mathrm{Pin}}] \rangle \neq 0$. $\Rightarrow$ Contradiction.

Another proof of Lemma 3

Monopole map

- For simplicity, we assume $b_1^\ell = \dim H_1(X;\ell) = 0$.
- Fix a reference $\mathrm{O}(2)$-connection A on E

$$\begin{aligned} \mu\colon \Gamma(S^+) \times \Omega(\ell \otimes i\mathbb{R}) &\to \Gamma(S^-) \times (\Omega^+ \oplus \Omega^0)(\ell \otimes i\mathbb{R}) \\ (a, \phi) \qquad &\mapsto (D_{A+a}\phi, F^+_{A+a} - q(\phi), d^* a) \end{aligned}$$

- μ is $\{\pm 1\}$-equivariant
- $\mu^{-1}(\text{ball}) \subset \text{ball}$

Finite dimensional approximation [Furuta,'95]

- Decompose $\mu = \mathcal{D} + \mathcal{Q}$ as $\mathcal{D}$: linear & $\mathcal{Q}$: quadratic
- Fix $\lambda \gg 1$.

$$V_\lambda = \mathrm{Span}\begin{pmatrix}\text{eigenspaces of } \mathcal{D}^*\mathcal{D} \\ \text{eigenvalues} < \lambda\end{pmatrix}$$

$$W_\lambda = \mathrm{Span}\begin{pmatrix}\text{eigenspaces of } \mathcal{D}\mathcal{D}^* \\ \text{eigenvalues} < \lambda\end{pmatrix}$$

- $p_\lambda \colon \Gamma(S^-) \times (\Omega^+ \oplus \Omega^0)(\ell \otimes i\mathbb{R}) \to W_\lambda$, L^2-projection
- Finite dim approx. $\boxed{f = \mathcal{D} + p_\lambda \mathcal{Q} \colon V_\lambda \to W_\lambda}$
- f is $\{\pm 1\}$-equivariant, proper

- f has the following form:

$$f\colon \tilde{\mathbb{R}}^{x+a} \oplus \mathbb{R}^{y} \to \tilde{\mathbb{R}}^{x} \oplus \mathbb{R}^{y+b}, \quad a = \operatorname{ind}_{\mathbb{R}} D_A,\ b = b_+^{\ell}(X)$$

$$\{\pm 1\} \text{ acts on } \begin{cases} \tilde{\mathbb{R}} \text{ by multiplication} \\ \mathbb{R} \text{ trivially} \end{cases}$$

$\tilde{\mathbb{R}}^{\bullet}$: spinor part, $\mathbb{R}^{\bullet}$: form part

- [Fact] $f|_{\{0\}\oplus\mathbb{R}^y}$ is a linear inclusion
- Suppose $b = b_+^{\ell}(X) = 0$. Consider the diagram

$$\begin{array}{ccc} \left(\tilde{\mathbb{R}}^{x+a} \oplus \mathbb{R}^{y}\right)^{+} & \xrightarrow{f^{+}} & \left(\tilde{\mathbb{R}}^{x} \oplus \mathbb{R}^{y}\right)^{+} \\ {\scriptstyle i_1}\uparrow & & {\scriptstyle i_2}\uparrow \\ (\mathbb{R}^{y})^{+} & \xrightarrow{\cong} & (\mathbb{R}^{y})^{+} \end{array}$$

$(\cdot)^{+}$ means the 1-point compactifications.

- Let $G = \{\pm 1\}$. Apply $\tilde{H}^*_G(\,\cdot\,;\mathbb{F}_2)$ to the diagram.

$$\begin{array}{ccc} \tilde{H}^*_G((\tilde{\mathbb{R}}^{x+a} \oplus \mathbb{R}^y)^+;\mathbb{F}_2) & \xleftarrow{(f^+)^*} & \tilde{H}^*_G((\tilde{\mathbb{R}}^{x} \oplus \mathbb{R}^y)^+;\mathbb{F}_2) \\ i_1^* \downarrow & & i_2^* \downarrow \\ \tilde{H}^*_G\left((\mathbb{R}^y)^+;\mathbb{F}_2\right) & \xleftarrow{\cong} & \tilde{H}^*_G\left((\mathbb{R}^y)^+;\mathbb{F}_2\right) \end{array}$$

- $\tilde{H}^*_G\left((\mathbb{R}^y)^+;\mathbb{F}_2\right) = \tilde{H}^*(BG;\mathbb{F}_2) = \tilde{H}^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) = \mathbb{F}_2[v],$ $(\deg v = 1)$
- $\mathrm{Im}\, i_1^* = \langle e(\tilde{\mathbb{R}}^{x+a})\rangle = \langle v^{x+a}\rangle$, $\mathrm{Im}\, i_2^* = \langle e(\tilde{\mathbb{R}}^{x})\rangle = \langle v^{x}\rangle$.
- Commutativity $\Rightarrow \mathrm{Im}\, i_1^* = \langle v^{x+a}\rangle \supset \mathrm{Im}\, i_2^* = \langle v^{x}\rangle$

$$\Rightarrow \quad \boxed{0 \geq a = \mathrm{ind}_{\mathbb{R}} D_A = \frac{1}{4}(-|w^2| + b_2^\ell)}$$

Definite 4-manifold with boundary

Recall:

- [Elkies]
  Q: Definite standard $\Leftrightarrow$
  $\forall$characteristic w, $0 \geq -|w^2| + \operatorname{rank} Q$. $(*)$
- For closed X, if Q_X, $Q_{X,\ell}$: definite $\Rightarrow (*)$.

- However if X has a boundary Y: $\mathbb{Z}HS^3$, even when Q_X or $Q_{X,\ell}$: definite, $(*)$ may not be true.
- Instead, we can estimate how false $(*)$ is by some quantities related with Y.

For $Q_X \to$ Froyshov invariant

For $Q_{X,\ell} \to \mathbb{Z}_2$-Froyshov invariant

Froyshov invariants

- Y: $\mathbb{Q}HS^3$ with Spin^c-structure
- [Froyshov invariant] $\delta^{\mathrm{U}(1)}(Y) = -h(Y) \in \mathbb{Q}$
  $\delta^{\mathrm{U}(1)}(Y)$: Manolescu's convention
  $h(Y)$: Froyshov, Kronheimer-Mrowka

Theorem [Froyshov]

- $\delta^{\mathrm{U}(1)}(Y_1 \# Y_2) = \delta^{\mathrm{U}(1)}(Y_1) + \delta^{\mathrm{U}(1)}(Y_2)$, $\delta^{\mathrm{U}(1)}(-Y) = -\delta^{\mathrm{U}(1)}(Y)$
- $\delta^{\mathrm{U}(1)}(Y)$ is a Spin^c-homology cobordism invariant.
- W: compact Spin^c 4-manifold s.t.
  $\partial W = Y_1 \cup \cdots \cup Y_k$, Y_i: $\mathbb{Q}HS^3$.

$$b_+(W) = 0 \quad \Rightarrow \quad \sum_{i=1}^{k} \delta^{\mathrm{U}(1)}(Y_i) \geq \frac{1}{8}(c_1(L)^2 + b_2(W))$$

where L is the determinant line bundle.

Corollary [Froyshov]

W: compact 4-manifold, $\partial W = Y_1 \cup \cdots \cup Y_k$, Y_i: $\mathbb{Z}HS^3$.
If $b_+(W) = 0$, for $\forall$ characteristic element w of Q_W

$$\sum_{i=1}^{k} \delta^{\mathrm{U}(1)}(Y_i) \geq \frac{1}{8}(-|w^2| + b_2(W))$$

$\mathbb{Z}_2$-Froyshov invariant

- Y: $\mathbb{Q}HS^3$ with Spin^c-structure

Theorem 3 [N,'16]

We can define a topological invariant $\delta^{\mathbb{Z}_2}(Y) \in \mathbb{Q}$

- $\delta^{\mathbb{Z}_2}(Y)$ is a homology cobordism invariant (for $\mathbb{Z}HS^3$).
- $\delta^{\mathbb{Z}_2}(Y) \leq 2\delta^{\mathrm{U}(1)}(Y)$
- W: cobordism from Y_0 to Y_1 with a $\mathbb{Z}$ bundle $\ell \to W$ s.t. $b_+^\ell(W) = 0$ & $\ell|_{Y_0}, \ell|_{Y_1}$: trivial
  For a Spin^{c-} structure on ℓ,

$$\delta^{\mathbb{Z}_2}(Y_1) \geq \delta^{\mathbb{Z}_2}(Y_0) + \frac{1}{4}(\tilde{c}_1(E)^2 + b_2^\ell(W))$$

[Remark] $\ell|_{Y_i}$: trivial $\Rightarrow$ Spin^{c-} str on W induces a Spin^c str on Y_i

Corollary [N,'16]

W: compact 4-manifold, $\partial W = Y_1 \cup \cdots \cup Y_k$, Y_i: $\mathbb{Z}HS^3$.
If $\ell \to W$ is a $\mathbb{Z}$ bundle with $w_1(\ell \otimes \mathbb{R})^2 = 0$ & $b_+^\ell(W) = 0$, for $\forall$characteristic element w of $Q_{W,\ell}$

$$\sum_{i=1}^{k} \delta^{\mathrm{U}(1)}(Y_i) \geq \frac{1}{2}\delta^{\mathbb{Z}_2}(Y_1 \# \cdots \# Y_K) \geq \frac{1}{8}(-|w^2| + b_2^\ell(W))$$

Conjecture

- $\delta^{\mathbb{Z}_2}(Y_1 \# Y_2) = \delta^{\mathbb{Z}_2}(Y_1) + \delta^{\mathbb{Z}_2}(Y_2)$
- $\delta^{\mathbb{Z}_2}(-Y) = -\delta^{\mathbb{Z}_2}(Y)$

A definition of $\delta^{\mathrm{U}(1)}(Y)$

- Let $X = \mathrm{SWF(Y)}$ be the Seiberg-Witten-Floer homotopy type constructed by Manolescu.
  Here we assume X as a space.
- $\mathrm{U}(1)$ acts on X & $X^{\mathrm{U}(1)} \cong (\mathbb{R}^s)^+$.
- Apply $\tilde{H}^*_{\mathrm{U}(1)}(\,\cdot\,;\mathbb{F})$ to the inclusion $i\colon X^{\mathrm{U}(1)} \to X$

$$i^*\colon \tilde{H}^*_{\mathrm{U}(1)}(X;\mathbb{F}) \to \tilde{H}^*_{\mathrm{U}(1)}(X^{\mathrm{U}(1)};\mathbb{F})$$

- Note $\tilde{H}^{*-s}_{\mathrm{U}(1)}(X^{\mathrm{U}(1)};\mathbb{F}) \cong \tilde{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{F}) \cong \mathbb{F}[u]$, $\deg u = 2$.
- Via this identification, $\exists d$, $\operatorname{Im} i^* = \langle u^d \rangle$.

$$\boxed{\delta^{\mathrm{U}(1)}(Y) = d + \text{(some grading shift)}}$$

Definition of $\delta^{\mathbb{Z}_2}(Y)$

- Since $X = \mathrm{SWF(Y)}$ is a $\mathrm{U}(1)$-space & $\mathrm{U}(1) \supset \{\pm 1\} = \mathbb{Z}_2$, $\mathbb{Z}_2$ acts on X & $X^{\mathbb{Z}_2} \cong (\mathbb{R}^s)^+$.
- Apply $\tilde{H}^*_{\mathbb{Z}_2}(\,\cdot\,;\mathbb{F}_2)$ to the inclusion $i\colon X^{\mathbb{Z}_2} \to X$

$$i^*\colon \tilde{H}^*_{\mathbb{Z}_2}(X;\mathbb{F}_2) \to \tilde{H}^*_{\mathbb{Z}_2}(X^{\mathbb{Z}_2};\mathbb{F}_2)$$

- Note $\tilde{H}^{*-s}_{\mathbb{Z}_2}(X^{\mathbb{Z}_2};\mathbb{F}_2) \cong \tilde{H}^*(\mathbb{RP}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[v]$, $\deg v = 1$.
- Via this identification, $\exists d_2$, $\operatorname{Im} i^* = \langle v^{d_2} \rangle$.

$$\boxed{\delta^{\mathbb{Z}_2}(Y) = d_2 + \text{(some grading shift)}}$$

Idea of the proof of Theorem 3

- Let W be the Spin^{c-} cobordism in Theorem 3.
- Let $X_i = \mathrm{SWF(Y_i)}$ $(i = 0, 1)$.
- $\mathrm{Pin}^-(2)$-monopole map on W induces a $\mathbb{Z}_2$-equivariant map

$$f \colon \Sigma^{\bullet} X_0 \to \Sigma^{\bullet} X_1$$

- We have a diagram

$$\begin{array}{ccc} \Sigma^{\bullet} X_0 & \xrightarrow{f} & \Sigma^{\bullet} X_1 \\ \uparrow & & \uparrow \\ (\Sigma^{\bullet} X_0)^{\mathbb{Z}_2} & \longrightarrow & (\Sigma^{\bullet} X_1)^{\mathbb{Z}_2} \end{array}$$

- Applying $\tilde{H}^*_{\mathbb{Z}_2}(\,\cdot\,; \mathbb{F}_2)$, we obtain the inequality in Theorem 3.

The outline of the proof of Theorem 2

Recall

Theorem 2

Let X be a closed connected ori. smooth 4-manifold. For any nontrivial $\mathbb{Z}$-bundle $\ell \to X$ s.t. $w_1(\ell \otimes \mathbb{R})^2 = w_2(X)$.

$$b_2^\ell(X) \geq \frac{10}{8}|\operatorname{sign}(X)| + 2,$$

where $b_2^\ell(X) = \operatorname{rank} H_2(X;\ell)$.

- $w_1(\lambda)^2 = w_2(X) \Rightarrow E = \underline{\mathbb{R}} \oplus (\ell \otimes \mathbb{R})$
  $\Rightarrow \mathrm{Spin}^{c-}$-structure on (X, E) has a larger symmetry

$$\mathcal{G}'_{\mathrm{Pin}} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{Pin}^-(2)).$$

\- For simplicity, assume $b_1^\ell(X) = 0$.

- By taking a finite dimensional approximation of the monopole map, we obtain a proper $\mathbb{Z}_4$-equivariant map

$$f\colon \tilde{\mathbb{R}}^y \oplus \mathbb{C}_1^{x+k} \to \tilde{\mathbb{R}}^{y+b} \oplus \mathbb{C}_1^x, \quad k = \mathrm{ind}_{\mathbb{C}} D_A,\ b = b_+^\ell(X)$$

$$\mathbb{Z}/4 \text{ acts on } \begin{cases} \mathbb{C}_1 \text{ by multiplication of } i \\ \tilde{\mathbb{R}} \text{ via } \mathbb{Z}/4 \to \mathbb{Z}/2 = \{\pm 1\} \curvearrowright \mathbb{R} \end{cases}$$

Here, $\mathbb{Z}_4$ is generated by the constant section

$$j \in \mathcal{G}' = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{Pin}^-(2)).$$

- By using the techniques of equivariant homotopy theory, e.g., tom Dieck's character formula, we can see that any proper $\mathbb{Z}_4$-map of the form,

$$f\colon \tilde{\mathbb{R}}^y \oplus \mathbb{C}_1^{x+k} \to \tilde{\mathbb{R}}^{y+b} \oplus \mathbb{C}_1^x,$$

  should satisfy $b > k$.
- That is,

$$b_+^\ell(X) \geq -\frac{1}{8}\,\mathrm{sign}(X) + 1.$$