Nonsmoothable group actions in dimension 4

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Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds

Nobuhiro Nakamura Nonsmoothable group actions

Preliminaries & Overview

on nonsmoothable group actions

Topological, locally linear and smooth actions

Let G be a finite group.

- X: C^0-manifold $\longrightarrow$ $G \curvearrowright X$: topological G-action.
- X: C^∞-manifold $\longrightarrow$ $G \curvearrowright X$: smooth G-action.

Definition

A topological G-action on X^n is called locally linear
if $\forall x \in X$, $\exists V_x$: a G_x-invariant nbd. of x s.t.
(G_x = the isotropy subgroup of x.)

- $V_x \underset{homeo}{\cong} \mathbb{R}^n$,
- G_x acts on $\mathbb{R}^n \cong V_x$ in a linear orthogonal way.

In general, smooth $\underset{\not\Leftarrow}{\Rightarrow}$ locally linear

Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds
Topological, locally linear and smooth actions
Nonsmoothable actions

3-classes of group actions

- G: finite group.
- X: a C^0-manifold.
- $X_\sigma \leftarrow$ a smooth structure σ specified.

$$\begin{aligned} Top(G,X) &:= \{\text{topological } G\text{-actions on } X\}/\sim_{homeo}, \\ LL(G,X) &:= \{\text{loc. lin. } G\text{-actions on } X\}/\sim_{homeo}, \\ C^\infty(G,X_\sigma) &:= \{\text{smooth } G\text{-actions on } X_\sigma\}/\sim_{diffeo}. \end{aligned}$$

$$Top(G,X) \supset LL(G,X) \overset{\varphi_\sigma}{\leftarrow} C^\infty(G,X_\sigma),$$

where φ_σ is the map forgetting the smooth structure.

Nobuhiro Nakamura
Nonsmoothable group actions

$$Top(G,X) \supset LL(G,X) \stackrel{\varphi_\sigma}{\leftarrow} C^\infty(G,X_\sigma),$$

Definition

A loc. lin. G-action on X is,

$$\text{smoothable w.r.t. } \sigma \stackrel{def}{\Leftrightarrow} \text{(its class)} \in \operatorname{im}\varphi_\sigma,$$

$$\text{nonsmoothable w.r.t. } \sigma \stackrel{def}{\Leftrightarrow} \text{(its class)} \notin \operatorname{im}\varphi_\sigma.$$

Remark

- φ_σ is surjective $\Leftrightarrow$ $\forall$ loc. lin. G-action is smoothable.
- φ_σ is not surjective $\Leftrightarrow$ $\exists$ nonsmoothable loc. lin. G-action.

Nonsmoothable actions

Question

When a locally linear G-action on X is smoothable?

Answer

No nonsmoothable loc. lin. action when $n = \dim X \leq 3$.

- $n \leq 2$: $\forall$top. G-action is smoothable.

$$C^\infty(G,X) \stackrel{\varphi_\sigma}{\to} LL(G,X) = Top(G,X) \quad \text{surjective}.$$

- $n = 3$: [Kwasik-Lee '88]

For closed X, smoothable $\Leftrightarrow$ locally linear

$$C^\infty(G,X) \stackrel{\varphi_\sigma}{\to} LL(G,X) \quad \text{surjective}.$$

Cf. In general, $LL(G,X) \subsetneq Top(G,X)$. $\leftarrow$ [Bing '52] etc.

$n = 4$

$\exists$ Many examples of nonsmoothable loc. lin. actions.

1. [Kwasik-Lee '88] $G = \mathbb{Z}_2 \curvearrowright X$: a closed smooth 4-manifold.
2. [Kwasik-Lawson '93]
   $G = \mathbb{Z}_p$ (p: prime)$\curvearrowright X$: contractible s.t.
   $\partial X = \Sigma(a, b, c)$: Brieskorn.
3. [Hambleton-Lee '95] $G = \mathbb{Z}_5 \curvearrowright X = \mathbb{C}P^2 \# \mathbb{C}P^2$.
4. [Bryan '98] $G = \mathbb{Z}_2 \curvearrowright X = K3$.
5. [Kiyono '04] $G = \mathbb{Z}_p$ (p: prime)$\curvearrowright X = \# S^2 \times S^2$.
6. [Liu-N '05-06] $G = \mathbb{Z}_p$ ($p = 3, 5, 7$)$\curvearrowright X = E(n)$.
7. [Chen-Kwasik '07] $\exists$family of symplectic exotic $K3$ s.t. $\forall$nontrivial odd order loc. lin. actions are nonsmoothable.
8. [N. '07] $G = \mathbb{Z}_2 \curvearrowright X = K3 \# K3$.

1.[Kwasik-Lee '88]

$\exists$nonsmoothable loc.lin.$\mathbb{Z}_{2^n}$-actions on a closed smooth 4-manifold.
(These actions are not orientation-preserving.)

Remark

In general, we need 2-steps to construct a nonsmoothable loc. lin. action:

- Existence: To construct loc. lin. actions concretely.
- Nonsmoothable: To prove the above actions do not satisfy conditions to be smooth.

In 1.[Kwasik-Lee '88],

- Existence $\leftarrow$ Surgery theory
- Nonsmoothable $\leftarrow$ Kirby-Siebenmann invariants

2.[Kwasik-Lawson '93]

$\exists$ nonsmoothable loc.lin. $\mathbb{Z}_p$-actions (p: prime) on a contractible smooth 4-manifold W s.t.

- orientation-preserving,
- $\partial W = \Sigma(a, b, c)$: Brieskorn homology 3-sphere,
- The action is free on $\partial W = \Sigma(a, b, c)$.

- Existence $\leftarrow$ [Edmonds '87]
- Nonsmoothable $\leftarrow$ Donaldson theory (Fintushel-Stern's invariants)

Cf. [Fukumoto-N.'07]

By using Fukumoto-Furuta's w-invariants, we can prove Nonsmoothability.

Edmonds-Ewing's construction of loc. lin. $\mathbb{Z}_p$-actions with discrete $X^{\mathbb{Z}_p}$

Let $G = \mathbb{Z}/p$, (p: prime). Suppose the following are given.

a) Fixed point data $\{(a_i, b_i)\}_{i=0,\dots,n+1}$.

b) G-invariant symmetric unimodular form

$$\Psi\colon V \times V \to \mathbb{Z}, \text{ where } V \text{ is a } \mathbb{Z}[G]\text{-module.}$$

[Edmonds '87], [Edmonds-Ewing '92]

Certain conditions on a) & b) $\Leftrightarrow$ $\exists$ loc. lin. G-action on $\exists X^4$ with $\pi_1(X) = 1$ realizing a), b).

The action is constructed by an equivariant handle construction.

Equivariant handle construction

Let $\mathbb{C}_k$ be the complex 1-dim. weight k representation of $\mathbb{Z}_p$.

$$B_{(a_0,b_0)} = D^4 \subset \mathbb{C}_{a_0} \oplus \mathbb{C}_{b_0} \leftarrow \text{0-handle}$$
$$D_{(a_i,b_i)} = D^2 \times D^2 \subset \mathbb{C}_{a_i} \oplus \mathbb{C}_{b_i}, (i = 1, \dots, n) \leftarrow G\text{-inv. 2-handles}$$

$\longrightarrow$ $G \curvearrowright X' := B_{(a_0,b_0)} \cup D_{(a_1,b_1)} \cup \cdots \cup D_{(a_n,b_n)} \cup ($ free 2-handles $)$.

Note

- $\partial X' =: \Sigma$ is an integral homology 3-sphere,
- $G \curvearrowright \partial X' = \Sigma$, freely.

Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds
Topological, locally linear and smooth actions
Nonsmoothable actions

Theorem (Edmonds-Ewing)

- *$\exists$ a contractible (C^0-)4-manifold W s.t. $\partial W = -\Sigma$.*
- *$\exists$ loc. lin. G-action on W extending $G \curvearrowright \Sigma = -\partial W$.*
- *$W^G = \{p_{n+1}\}$. The type of p_{n+1} is (a_{n+1}, b_{n+1}).*

The proof uses Freedman's theory.

Thus, $G \curvearrowright X := X' \cup W$ is a required action.

Nobuhiro Nakamura
Nonsmoothable group actions

3.[Hambleton-Lee '95]

$\exists$ nonsmoothable loc. lin. $\mathbb{Z}/5$-action on $\mathbb{C}P^2 \# \mathbb{C}P^2$

- Existence $\leftarrow$ [Edmonds-Ewing '92]
- Nonsmoothable $\leftarrow$ Donaldson theory
  (G-equivariant "Theorem A")

Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds
Topological, locally linear and smooth actions
Nonsmoothable actions

4.[Bryan '98]

$\exists$ nonsmoothable loc. lin. $\mathbb{Z}/2$-action on $K3$

- Existence $\leftarrow$ [Edmonds-Ewing '92]
- Nonsmoothable $\leftarrow$ Seiberg-Witten (refined 10/8-inequality)

The refined 10/8-inequality [Bryan]

If $\exists$ smooth $\mathbb{Z}/2^n$-action on a spin manifold X with some conditions,

$$\Rightarrow \ b_2(X) \geq \frac{10}{8}|\operatorname{Sign}(X)| + 2 + 2n.$$

Nobuhiro Nakamura
Nonsmoothable group actions

5.[Kiyono '04]

$\exists$ nonsmoothable loc. lin. $\mathbb{Z}_p$-actions (p: prime) on $\#S^2 \times S^2$.

- Existence $\leftarrow$ [Edmonds-Ewing '92]
- Nonsmoothable $\leftarrow$ Seiberg-Witten (G-equiv. 10/8-inequality)

  If $G \curvearrowright X$: spin $\Rightarrow$ X/G should satisfy "10/8-inequality".

Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds
Topological, locally linear and smooth actions
Nonsmoothable actions

6.[Liu-N. '05-'06]

$\exists$ nonsmoothable loc. lin. $\mathbb{Z}_p$-actions ($p = 3, 5, 7$) on $E(n)$.

- Existence $\leftarrow$ [Edmonds-Ewing '92]
- Nonsmoothable $\leftarrow$ Seiberg-Witten invariants
  Mod p vanishing theorem under $\mathbb{Z}_p$-actions

Nobuhiro Nakamura
Nonsmoothable group actions

Mod p vanishing theorem [Fang]([N.])

Some conditions on fixed point data

$$\Rightarrow \ \mathrm{SW}_X(c) \equiv 0 \mod p.$$

On the other hand,

$$\mathrm{SW}_{E(n)}(c_{\mathrm{spin}}) = \pm \binom{n-2}{\frac{n-2}{2}}.$$

$\therefore$ If $SW_{E(n)}(c_{\mathrm{spin}}) \not\equiv 0 \mod p \Rightarrow$ Not "Some conditions".

Remark

- Loc. lin. actions in 2.3.4.5. are nonsmoothable for $\forall$smooth structures.
  2. [Kwasik-Lawson '93] $G = \mathbb{Z}_p$ (p: prime)$\curvearrowright W$: contractible s.t. $\partial W = \Sigma(a,b,c)$: Brieskorn.
  3. [Hambleton-Lee '95] $G = \mathbb{Z}_5 \curvearrowright X = \mathbb{C}P^2 \# \mathbb{C}P^2$.
  4. [Bryan '98] $G = \mathbb{Z}_2 \curvearrowright X = K3$.
  5. [Kiyono '04] $G = \mathbb{Z}_p$ (p: prime)$\curvearrowright X = \# S^2 \times S^2$.
- For example, 4.5. use "10/8-inequality" on b_2 & the signature which are not depending on smooth structures.
- In other words, put $\Sigma := \{$ smooth structures on X $\}$, $\Rightarrow$ 2.3.4.5. claim

$$\left(LL(G,X) \setminus \bigcup_{\sigma\in\Sigma} \mathrm{im}\,\varphi_\sigma \right) \neq \emptyset.$$

- In 6.([Liu-N '05-06] $G = \mathbb{Z}_p \curvearrowright E(n)$), we need to check SW-inv. for each smooth structure.
  $\longrightarrow$ Our examples would be subtle???

- The following may happen:

$$\exists \sigma, \sigma' \in \Sigma, \ \text{(our example)} \ \begin{array}{l} \in \operatorname{im} \varphi_\sigma \\ \notin \operatorname{im} \varphi_{\sigma'} \end{array}$$

In fact, our examples are nonsmoothable w.r.t. infinitely many smooth strutures including the standard one.

7.[Chen-Kwasik '07]
$\exists$a family of symplectic exotic $K3$'s s.t.
$\forall$nontrivial odd order loc. lin. actions are nonsmoothable.

- We have several smooth $\mathbb{Z}_p$-actions (p: odd) on the standard K3.
- As loc. lin. actions, these are nonsmoothable for above exotic structures.

- Nonsmoothable
  - $\forall$diffeo. should preserve SW basic classes
  - [Taubes] SW $\Rightarrow$ Gr

  $\Bigg\} \Rightarrow$ A strong constraint on the action on $H^2(X;\mathbb{Z})$.

8.[N. '07]

∃loc. lin. $\mathbb{Z}_2$-action on $K3\#K3$ which is nonsmoothable w.r.t. ∀smooth structure.

- Existence ← [Edmonds-Ewing '92]
- Nonsmoothable ← Bauer-Furuta invariants
  A vanishing theorem under $\mathbb{Z}_2$-actions

- Bauer-Furuta invariant is a stable cohomotopy refinement of SW invariant.
- $SW_{K3\#K3} \equiv 0$, but
  $BF_{K3\#K3}(c_{spin}) \neq 0$ by [Furuta-Kametani-Minami '01].
- [N.'07] A Bauer-Furuta version of the mod 2 vanishing theorem holds.

9.[N.'07]

∃loc. lin. $\mathbb{Z}_2$-action on $K3$ s.t.

- nonsmoothable for ∀smooth structures,
- $X^{\mathbb{Z}_2}$: discrete,
- $b_+^{\mathbb{Z}_2} := \dim H^+(X;\mathbb{R})^{\mathbb{Z}_2} = 3$.

- Existence ← [Edmonds-Ewing '92]
- Nonsmoothable ← Mod 2 vanishing theorem for SW.

Cf. [Bryan, '98]

For smooth $\mathbb{Z}_2$-actions on $K3$,

- $X^{\mathbb{Z}_2}$: discrete & $b_+^{\mathbb{Z}_2} = 3$,
  or
- $\dim X^{\mathbb{Z}_2} = 2$ & $b_+^{\mathbb{Z}_2} = 1$.

Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds

A joint work with Y. Fukumoto

Recall:

[Kwasik-Lawson '93]

$\exists$ nonsmoothable loc.lin. $\mathbb{Z}_p$-actions (p: prime) on a contractible smooth 4-manifold W s.t.

- orientation-preserving,
- $\partial W = \Sigma(a, b, c)$: Brieskorn homology 3-sphere,
- The action is free on $\partial W = \Sigma(a, b, c)$.

- To prove nonsmoothability, [Kwasik-Lawson] uses Donaldson theory. $\longrightarrow$ Fintushel-Stern's invariants
- We ([Fukumoto-N.]) give an alternative proof by using Seiberg-Witten theory. $\longrightarrow$ w-invariants

Background on gauge theory

- [Early '80s–] Donaldson theory $\longleftarrow$ ASD equation
- ['94–] Seiberg-Witten theory $\longleftarrow$ Seiberg-Witten equations
- Many results obtained from Donaldson theory can be proved by using Seiberg-Witten theory, too.

Witten's conjecture('94)

Donaldson invariants are equivalent to Seiberg-Witten invariants.

Theorem (Feehan-Leness '06)

Witten's conjecture is true when X is of simple type and $b_+ \geq 2$.

Question

Is Donaldson theory strictly equivalent to Seiberg-Witten theory in mathematics?

$\longrightarrow$ Probably, NO.

- Bauer-Furuta invariants. $\longrightarrow$ No stable cohomotopy invariant in Donaldson theory, at present.
- [Sasahira '06] 2-torsion instanton invariants.
  $\longrightarrow$ No corresponding invariant in Seiberg-Witten theory.
- [Fukumoto '07] There are differences between consequences from Fintushel-Stern's invariants & w-invariants.

Now, we are seeking such a difference in our case.

Our result

- Let a, b, c be mutually coprime integers.
- $\Sigma(a, b, c)$: Brieskorn 3-sphere.
- Let $\mathbb{Z}_p$ act on $\Sigma(a, b, c)$ as a part of the Seifert circle action.
- Suppose p is coprime to a, b, c. $\Rightarrow$ The $\mathbb{Z}_p$-action is free.

Theorem (Fukumoto-N. '07)

$\exists(a, b, c)$, $\exists p$ and $\exists$ smooth contractible 4-mfd W s.t.

- *$\partial W = \Sigma(a, b, c)$,*
- *the $\mathbb{Z}_p$-action on $\Sigma(a, b, c)$ extends locally linearly over W,*
- *but no such smooth action.*

Examples of (a, b, c) & p

- $p = 3$, $(a, b, c) = (2, 11, 53)$, $(5, 13, 14)$, $(5, 16, 17) \cdots$
- $p = 5$, $(a, b, c) = (11, 31, 32)$. $(3, 13, 14)$, $(3, 16, 17) \cdots$
- $p = 7$, $(a, b, c) = (3, 19, 20)$, $(3, 22, 23)$, $(3, 40, 41) \cdots$

Remark

- At present, our result is completely contained in the result by [Kwasik-Lawson].
- That is, we do not find the difference between the consequences from w-invariants & Fintushel-Stern's invariants in this situation.

Proof of Theorem

Existence of W & loc. lin. actions

- [Casson-Harer] etc. $\longrightarrow$ Existence of smooth W.
- [Edmonds], [Kwasik-Lawson] $\longrightarrow$ Existence of loc. lin. actions.

Nonsmoothability

- Suppose the $\mathbb{Z}_p$-action on $\Sigma(a,b,c)$ extends smoothly over W with one fixed point x.

$\Rightarrow$ For some $\mathbb{Z}_p$-inv. nbd N of x,

$$U := (W \setminus N)/\mathbb{Z}_p,$$

gives a smooth homology cobordism between $Q := \Sigma(a,b,c)/\mathbb{Z}_p$ and some lens space $L(p,q)$.

$\rightarrow$ Seiberg-Witten theory will give a constraint on such smooth cobordisms.

- For $\Sigma(a,b,c)$, $\exists V$-line bundle $\pi\colon L \to Z(\cong S^2)$ s.t.

$$\Sigma(a,b,c) \cong S(L). \leftarrow \text{ the unit sphere bundle of } L$$

- Then,

$$Q = \Sigma(a,b,c)/\mathbb{Z}_p \cong S(L^p).$$

- Let $V := D(L^{-p})$. $\leftarrow$ the unit disk bundle of L^{-p}
  Let us consider the V-manifold,

$$X := V \cup U \cup cL(p,q),$$

where $cL(p,q)$ is the cone of $L(p,q)$.

- Fix a Spin^c-structure c on X. The Seiberg-Witten moduli for c is given as,

$$\mathcal{M}_c := \{\text{solutions to Seiberg-Witten equations}\}/\mathcal{G},$$

where $\mathcal{G}$ is the gauge transformation group.

- Let $d(c)$ be the virtual dimension of $\mathcal{M}_c$.
$d(c)$ is essentially the w-invariant of U.

Proposition

For $\forall\,\mathrm{Spin}^c$-structure c, $d(c) \leq 0$.

Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds
Background on gauge theory
Our result
Proof of Theorem

Proof of Proposition

- X: negative definite & $b_1(X) = 0$.
$\Rightarrow d(c)$ is odd & $\exists!$reducible $\in \mathcal{M}_c$.
- Suppose $d(c) > 0$.
- $\mathcal{M}_c$ is a compact space.
- By perturbing the SW equations,
$\Rightarrow \mathcal{M}_c \setminus \{\text{reducible}\}$ becomes a $d(c)$-dim. mfd.
- Remove an open nbd N of the reducible.
$\Rightarrow \bar{\mathcal{M}}_c := \mathcal{M}_c \setminus N$ is a compact $d(c)$-mfd s.t. $\partial\bar{\mathcal{M}}_c = \mathbb{C}\mathrm{P}^k$,
where $k = (d(c) - 1)/2$.
- In fact,

$$\bar{\mathcal{M}}_c \subset \mathcal{B}^* \simeq \mathbb{C}\mathrm{P}^\infty \;\&\; \exists\alpha \in H^{2k}(\mathcal{B}^*) \text{ s.t. } \langle\alpha, [\mathbb{C}\mathrm{P}^k]\rangle \neq 0.$$

$\rightarrow$ A contradiction. $\therefore d(c) \leq 0$. □

Nobuhiro Nakamura
Nonsmoothable group actions

Calculation of $d(c)$

- $\exists c_0$: the canonical Spin^c-structure associated to the complex structure of $V = D(L^{-p})$.
- c_m: the Spin^c-structure obtained by twisting c_0 by $\pi^* L^m$.
- $d(c_m)$ can be calculated by Kawasaki's V-index theorem.

Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds
Background on gauge theory
Our result
Proof of Theorem

$$
\begin{aligned}
d(c_m) =& \frac{1}{4}\Bigg[-\frac{1}{p\alpha}(f+p+2m)^2 + 1 \\
& - \sum_{i=1}^{n} \frac{1}{a_i} \sum_{l=1}^{a_i-1} \Bigg\{ \cot\left(\frac{\pi l}{a_i}\right) \cot\left(\frac{\pi p b_i l}{a_i}\right) \\
& + 2\cos\left(\frac{\pi(1+pb_i+2mb_i)l}{a_i}\right) \csc\left(\frac{\pi l}{a_i}\right) \csc\left(\frac{\pi p b_i l}{a_i}\right) \Bigg\} \\
& - \frac{1}{p} \sum_{l=1}^{p-1} \Bigg\{ \cot\left(\frac{\pi l}{p}\right) \cot\left(\frac{\pi q l}{p}\right) \\
& + 2\cos\left(\frac{\pi(1+q+2m')l}{p}\right) \csc\left(\frac{\pi l}{p}\right) \csc\left(\frac{\pi q l}{p}\right) \Bigg\} \Bigg] - 1.
\end{aligned}
$$

Please see the abstract for detail!

If one of the following holds:

- $\exists c_m$ s.t. $d(c_m) \notin \mathbb{Z}$,
- $\exists c_m$ s.t. $d(c_m) > 0$,

$\Rightarrow$ NO smooth homology cobordism between Q and $L(p,q)$

$\Rightarrow$ NO smooth extension of the $\mathbb{Z}_p$-action on $\Sigma(a,b,c)$ over W.

Preliminaries & Overview
Nonsmoothable $\mathbb{Z}_p$-actions on contractible 4-manifolds
Background on gauge theory
Our result
Proof of Theorem

Remark

- Fintushel-Stern's invariant $R(k)$ is defined as

$$R(k) := \text{ The virtual dimension of the ASD-moduli on } X,$$

  where k is the instanton number.
- If k is small enough
  $\Rightarrow$ The ASD-moduli is compact
  $\Rightarrow$ $R(k) \leq 0$.

Nobuhiro Nakamura
Nonsmoothable group actions