Nonsmoothable involutions on $K3$ and $K3\#K3$

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January 28, 2009

Main Results

Main Theorem

1. *$\exists$Nonsmoothable locally linear $\mathbb{Z}_2$-action on $X = K3$.*
2. *$\exists$Nonsmoothable locally linear $\mathbb{Z}_2$-action on $X = K3\#K3$.*

Remark

The above actions are nonsmoothable w.r.t. $\forall$smooth structures on X.

Strategy for proof

The proof is divided into 2 steps:

1. Give constraints on smooth actions
   (1) $K3 \longrightarrow$ Rohlin's theorem
   (2) $K3\#K3 \longrightarrow$ Bauer-Furuta invariants
2. Consruct loc. lin. actions which violate the constraints.
   $\longrightarrow$ [Edmonds-Ewing'92]

Remark

- [N.] used the Seiberg-Witten invariants for 1(1).
- The referee of the journal pointed out that Rohlin's theorem is enough for 1(1)!!

Known results

Nonsmoothable loc. lin. involutions

- [Kwasik-Lee'88] $\mathbb{Z}_2 \curvearrowright S^4$. (Not orientation-preserving.)
- [Kwasik-Lawson'93] $\mathbb{Z}_2 \curvearrowright W$: contractible s.t. $\partial W =$ Brieskorn.
- [Bryan'98] $\mathbb{Z}_2 \curvearrowright K3$.

Cf. Corks [Akbulut-Yasui]...

Nonsmoothable loc. lin. actions on $K3$

- [Bryan'98] as above.
- [Liu-N.'07] $\mathbb{Z}_p \curvearrowright K3$, $(p = 3, 5, 7)$.
- [Chen-Kwasik'07] $\mathbb{Z}_p \curvearrowright$ exotic $K3$, (p: prime, ≥ 7).
- [Kiyono'08] $\mathbb{Z}_p \curvearrowright K3$, ($p$: large prime).

Contents

Nonsmoothable involution on $K3$

Main Theorem 1

There exists loc. lin. $\mathbb{Z}_2$-action on $X = K3$ s.t.

1. $X^{\mathbb{Z}_2}$: discrete & $\#X^{\mathbb{Z}_2} = 8$,
2. $b_+(X/\mathbb{Z}_2) = 3$,
3. nonsmoothable for any smooth structure on X.

Remark

- [Bryan'98] For smooth involutions on $X = K3$,

$$X^{\mathbb{Z}_2} : \text{discrete} \Rightarrow \#X^{\mathbb{Z}_2} = 8 \ \& \ b_+(X/\mathbb{Z}_2) = 3$$

- The action in Main Theorem 1 has the same fixed point data & action on $K3$ lattice as "Nikulin involution".

Constraint on smooth involutions 1 (Rohlin's theorem)

Suppose

- X: smooth, closed, oriented, spin 4-manifold, $\pi_1 X = 1$.
- $\mathbb{Z}_2 \curvearrowright X$ smoothly, ori. preserving

[Atiyah-Bott]

$X^{\mathbb{Z}_2}$: discrete $\Leftrightarrow$ The $\mathbb{Z}_2$-action lifts to the spin structure.

$\Rightarrow X/\mathbb{Z}_2$ is a spin V-manifold.

- Quotient singularities = cones of $\mathbb{R}P^3$.

Constraint on smooth involution 1 (Rohlin's theorem)

- There are 2 equivalence classes $s_\pm$ of spin structures on $\mathbb{R}P^3$.
- Let $\tilde{s}_\pm$ be the unique spin str. on the disk bundle $D_\pm$ over S^2 of degree ± 2. $\Rightarrow s_\pm = \tilde{s}_\pm|_{\partial D_\pm}$.
- Define the spin type of a fixed point by the spin str. on $\mathbb{R}P^3$ induced from $X/\mathbb{Z}_2$.
- Let $n_\pm = \#(\text{fixed points with } s_\pm)$. $\Rightarrow \# X^{\mathbb{Z}_2} = n_+ + n_-$.

Rohlin's theorem

$$\sigma(X/\mathbb{Z}_2) \equiv n_+ - n_- \mod 16$$

- Note $\sigma(X/\mathbb{Z}_2) = \frac{1}{2}\sigma(X)$.

$$\rightarrow \boxed{\frac{1}{2}\sigma(X) \equiv n_+ - n_- \mod 16}$$

Corollary

If $X = K3$ & $\#X^{\mathbb{Z}_2} = 8$, then $(n_+, n_-) = (8, 0)$ or $(0, 8)$.

Remark

$\sigma(X/\mathbb{Z}_2)$, n_+ and n_- do not depend on smooth structures.

They are invariants of loc. lin. $\mathbb{Z}_2$-actions on topological spin 4-manifolds X with discrete $X^{\mathbb{Z}_2}$.

Construction of loc. lin. $\mathbb{Z}_2$-actions

Theorem (Edmonds-Ewing '92)

$\Psi \colon V \times V \to \mathbb{Z}$ a $\mathbb{Z}_2$-inv. symm. unimodular even form s.t.

1. *As a $\mathbb{Z}[\mathbb{Z}_2]$-module, $V \cong T \oplus F$,*

$$\text{where } T \cong n\mathbb{Z} \leftarrow \text{a trivial } \mathbb{Z}[\mathbb{Z}_2]\text{-module}$$
$$F \cong k\mathbb{Z}[\mathbb{Z}_2] \leftarrow \text{a free } \mathbb{Z}[\mathbb{Z}_2]\text{-module}$$

2. *$\forall v \in V, \Psi(gv, v) \equiv 0 \mod 2$.*
3. *G-signature formula $\mathrm{Sign}(g, (V, \Psi)) = 0$.*

$\Rightarrow \exists$loc. lin $\mathbb{Z}_2$-action on a simply-connected 4-manifold X s.t.

- *Its intersection form $= \Psi$,*
- *$\#X^{\mathbb{Z}_2} = n + 2$.*

Remark

Since Ψ is supposed even, the homeotype of X is unique

Idea of Proof $\rightarrow$ Equivariant handle construction on

$$\Psi\colon V \times V \to \mathbb{Z}, \quad V = T \oplus F.$$

A unit 4-ball $B_0 \subset \mathbb{C}^2 \curvearrowleft \{\pm 1\}$

$T \leftrightarrow H_1, \ldots, H_n:$ copies of $D^2 \times D^2 \subset \mathbb{C}^2 \curvearrowleft \{\pm 1\}$

$F \leftrightarrow$ free 2-handles

Note: $B_0^{\mathbb{Z}_2} = \{0\}$, $(D^2 \times D^2)^{\mathbb{Z}_2} = \{0\}$.

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Step 1.

Represent Ψ by a $\mathbb{Z}_2$-invariant framed link L in ∂B_0.

Step2.

Attach $H_1, \ldots, H_n$ and free handles to B_0 equivariantly along L.

$$\longrightarrow \mathbb{Z}_2 \curvearrowright B_0 \cup H_1 \cup \cdots \cup H_n \cup (\text{free handles}).$$

The $\mathbb{Z}_2$-action on $B_0 \cup H_1 \cup \cdots \cup H_n \cup (\text{free handles})$ is smooth.

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Step3. Note

- $\Sigma := \partial(B_0 \cup H_1 \cup \cdots \cup H_n \cup (\text{free handles}))$ is a $\mathbb{Z}$-homology 3-sphere,
- $\mathbb{Z}_2 \curvearrowright \Sigma$: free.

Theorem ([EE])

Under the above assumptions, $\exists$loc. lin $\mathbb{Z}_2$-action on $\exists W^4$ s.t.

- *W: contractible & $\partial W \cong \Sigma$,*
- *$(\mathbb{Z}_2 \curvearrowright \partial W) \cong (\mathbb{Z}_2 \curvearrowright \Sigma)$,*
- *$W^{\mathbb{Z}_2} = \{1 \text{ point}\}$.*

$\to\exists$ Loc. lin. involution:

$$\mathbb{Z}_2 \curvearrowright X = (B_0 \cup H_1 \cup \cdots \cup H_n \cup (\text{free handles})) \cup_\Sigma W$$

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$\mathbb{Z}_2 \curvearrowright X = B_0 \cup H_1 \cup \cdots \cup H_n \cup (\text{free handles}) \cup W$, locally linear.

- Each of $B_0, H_1, \ldots, H_n, W$ has one fixed point: $P, Q_1, \ldots, Q_n, P'$.

$$\begin{array}{cccc} L = & K_1 \cup \cdots \cup & K_n \cup \cdots, & \\ & \updownarrow r_1 & \updownarrow r_n & \\ & H_1 & H_n & \end{array}$$

- $r_i :=$ (the framing of K_i).

How about spin types of fixed points?

Proposition

Suppose K_i bounds a $\mathbb{Z}_2$-invariant embedded disk in B_0.

$r_i \equiv 2 \mod 4 \Leftrightarrow$ *P and Q_i have same spin types.*

$r_i \equiv 0 \mod 4 \Leftrightarrow$ *P and Q_i have different spin types.*

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Construction of a nonsmoothable involution on $K3$

$X = K3 \Rightarrow \Psi_X \cong 2E_8 \oplus 3H$.
Define $\mathbb{Z}_2$-action on $2E_8 \oplus 3H$ as follows:

- $\mathbb{Z}_2 \curvearrowright E_8 \oplus E_8$: Permutation
- $\mathbb{Z}_2 \curvearrowright 3H$: Trivial

Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} \leftrightarrow \text{indefinite, even, unimodular} \cong 3H$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Let $p\colon S^3 \to S^2$ be the Hopf fibration. Put $L_T = p^{-1}(6 \text{ points})$
$\Rightarrow L_T$ represents A.

Let L_{E_8} be a framed link which represents E_8.
$\Rightarrow L := L_T \sqcup (2 \text{ copies of } L_{E_8})$ represents $3H \oplus 2E_8$.
Note that each component of L_T bounds a $\mathbb{Z}_2$-invariant embedded disk.

$\rightarrow$ $\exists$ a loc. lin. involution on $X = K3$ with $\begin{cases} \#X^{\mathbb{Z}_2} = 8 \\ (n_+, n_-) = (4, 4) \end{cases}$

- Recall [Rohlin's theorem] implies

$$\left.\begin{array}{c}\mathbb{Z}_2 \curvearrowright X = K3 \text{ smoothly} \\ \#X^{\mathbb{Z}_2} = 8\end{array}\right\} \Rightarrow (n_+, n_-) = (8,0) \text{ or } (0,8)$$

- Thus, the above loc. lin. action with $\left\{\begin{array}{c}\#X^{\mathbb{Z}_2} = 8 \\ (n_+, n_-) = (4,4)\end{array}\right\}$ is

Nonsmoothable!!

Nonsmoothable involution on $K3\#K3$

Main Theorem 2

There exists loc. lin. $\mathbb{Z}_2$-action on $X = K3\#K3$ s.t.

1. $X^{\mathbb{Z}_2}$: discrete & $\#X^{\mathbb{Z}_2} = 10$,
2. $b_+(X/\mathbb{Z}_2) = 5$,
3. nonsmoothable for any smooth structure on X.

Constraint on smoooth involutions 2 (Bauer-Furuta invariants)

- X: smooth, closed, oriented 4-manifold
- c: Spin^c-structure on X

Suppose $\mathbb{Z}_2 \curvearrowright (X, c)$ smoothly.

$\Rightarrow \mathrm{ind}_{\mathbb{Z}_2}\,\mathsf{Dirac} = k_+ \cdot 1 + k_- \cdot t \in R(\mathbb{Z}_2) \cong \mathbb{Z}[t]/(t^2 - 1)$.

Theorem (Vanishing theorem of Bauer-Furuta invariants, [N.'08])

Suppose

1. $b_1(X) = 0,\ b_+(X) \geq 2,\ b_+(X/\mathbb{Z}_2) \geq 1$.
2. $d(c) := 2(k_+ + k_-) - (1 + b_+(X)) = 1$.
3. $2k_\pm < 1 + b_+(X/\mathbb{Z}_2)$.
4. $b_+(X) - b_+(X/\mathbb{Z}_2)$ *is odd.*

Then the Bauer-Furuta invariant of (X, c) *is* 0: $\mathrm{BF}(c) = 0$.

Remark

- $d(c)$ *is the virtual dimension of the SW-moduli for* c.
- *When* $d(c) = 1$,
  - $k_+ + k_-$*: even* $\Rightarrow \mathrm{BF}(c) \in \mathbb{Z}/2$.
  - $k_+ + k_-$*: odd* $\Rightarrow \mathrm{BF}(c) \in \{0\}$.

Application to spin manifolds

- Suppose $\mathbb{Z}_2 \curvearrowright (X, spin)$ smoothly. $\Rightarrow X^{\mathbb{Z}_2}$: discrete
- G-spin theorem $\Rightarrow 2k_\pm = \dfrac{1}{4}\left(\dfrac{-\sigma(X)}{2} \pm (n_+ - n_-)\right)$.

Corollary

$$\left.\begin{array}{l} b_1(X) = 0, b_+(X) \geq 2, b_+(X/\mathbb{Z}_2) \geq 1 \\ d(spin) = 1 \\ b_+(X) - b_+(X/\mathbb{Z}_2) \text{ is odd} \\ \mathrm{BF}(spin) = 1 \in \mathbb{Z}_2 \end{array}\right\} \Rightarrow \begin{array}{c} \dfrac{1}{4}\left(\dfrac{-\sigma(X)}{2} + |n_+ - n_-|\right) \\ \geq 1 + b_+(X/\mathbb{Z}_2) \end{array}$$

Fact (Furuta-Kametani-Minami '01)

$$X = \text{homotopy } K3\#K3 \Rightarrow d(spin) = 1 \ \& \ \mathrm{BF}(spin) = 1.$$

Construction of a nonsmoothable involution on $K3\#K3$

$X = K3\#K3 \Rightarrow \Psi_X \cong 4E_8 \oplus 6H$.

Define $\mathbb{Z}_2$-action on $4E_8 \oplus 6H$ as follows:

- $\mathbb{Z}_2 \curvearrowright 2E_8 \oplus 2E_8$: Permutation
- $\mathbb{Z}_2 \curvearrowright H \oplus H$: Permutation
- $\mathbb{Z}_2 \curvearrowright 4H$: Trivial

Let

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} \leftrightarrow \text{indefinite, even, unimodular} \cong 4H$$

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Let $p\colon S^3 \to S^2$ be the Hopf fibration. Put $L_T = p^{-1}(8 \text{ points})$
$\Rightarrow L_T$ represents B.

$\to$ $\exists$ a loc. lin. involution on $X = K3\#K3$ with $\begin{cases} \#X^{\mathbb{Z}_2} = 10 \\ b_+(X/\mathbb{Z}_2) = 5 \\ (n_+, n_-) = (5,5) \end{cases}$

If the action is smoothed

$$\begin{aligned} \Rightarrow |n_+ - n_-| \geq & \frac{\sigma(X)}{2} + 4(1 + b_+(X/\mathbb{Z}_2)) \\ = & -16 + 4(1+5) = 8. \end{aligned}$$

$\to$Contradiction to $(n_+, n_-) = (5,5)$

$\to$Nonsmoothable!!

(*Cf.* Rohlin $\Rightarrow \sigma(X)/2 = -16 \equiv n_+ - n_- \mod 16$.)

The proof of the vanishing theorem

Bauer-Furuta invariants

- M. Furuta introduced a finite dimensional model describing the Seiberg-Witten moduli $\mathcal{M}_{X,c}$ of (X, c):

$$\exists f\colon V \to W,\ S^1\text{-equivariant, proper}$$

$$\text{where} \begin{cases} V = \mathbb{C}^a \oplus \mathbb{R}^b, W = \mathbb{C}^x \oplus \mathbb{R}^y \\ S^1 \curvearrowright \mathbb{C}^\bullet \text{ by multiplication} \\ S^1 \curvearrowright \mathbb{R}^\bullet \text{ trivially} \end{cases}$$

s.t. $f^{-1}(0)/S^1 \cong \mathcal{M}_{X,c}$.

Theorem & Definition (Bauer-Furuta'04)

The stable homotopy class of f does not depend on parameters.
The Bauer-Furuta invariant is defined as

$$\mathrm{BF}(c) := [f] \in \{S^V, S^W\}^{S^1}.$$

- If $\mathbb{Z}_2 \curvearrowright (X, c) \Rightarrow f\colon V \to W$ is $\mathbb{Z}_2 \times S^1$-equivariant.
  $\Rightarrow \mathbb{Z}_2$-equiv. BF invariant can be defined as

$$\mathrm{BF}^{\mathbb{Z}_2}(c) = [f] \in \{S^V, S^W\}^{\mathbb{Z}_2 \times S^1}.$$

Relation

$$\phi\colon \{S^V, S^W\}^{\mathbb{Z}_2\times S^1} \to \{S^V, S^W\}^{S^1}$$ ←forgetting the $\mathbb{Z}_2$-action

$$\boxed{\mathrm{BF}(c) = \phi(\mathrm{BF}^{\mathbb{Z}_2}(c))}$$

The idea of the proof of the vanishing theorem

- Under the assumptions of theorem, we prove ϕ is 0 map.
  →Use equivariant obstruction theory on Bredon cohomology
- The proof is inspired by [Bauer '08].

Sketch of the proof

Let us consider a special case:

$$\begin{matrix} V = \mathbb{C}_+^2 \oplus \mathbb{C}_-^2 \\ W = \mathbb{R}_+^5 \oplus \mathbb{R}_- \end{matrix} \quad \text{where} \quad \begin{cases} \mathbb{Z}_2 \curvearrowright \mathbb{C}_+, \mathbb{R}_+, \ \text{trivially} \\ \mathbb{Z}_2 = \{\pm 1\} \curvearrowright \mathbb{C}_-, \mathbb{R}_- \ \text{multiplication} \end{cases}$$

Lemma

Let $P(V)$ be the complex projective space of V with the induced $\mathbb{Z}_2$-action.

$$\{S^V, S^W\}^{S^1} \cong H^6(P(V); \pi_6(S^5))$$
$$\{S^V, S^W\}^{\mathbb{Z}_2\times S^1} \cong H^6_{\mathbb{Z}_2}(P(V); \underline{\pi}_6(S^5))$$ ← *Bredon cohomology*

- Fix a $\mathbb{Z}_2$-equivariant cell comlpex structure on $P(V)$.
  $\Rightarrow$ Its chain cpx C_* is a $\mathbb{Z}[\mathbb{Z}_2]$-module.
- Let us consider the diagram:

$$\begin{array}{ccccccc}
0 & \longleftarrow & H^6_{\mathbb{Z}_2}(P(V);\underline{\pi}_6(S^5)) & \longleftarrow & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(C_6,\mathbb{Z}_2) & \xleftarrow{\delta} & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(C_5,\mathbb{Z}_2) \\
 & & \downarrow \phi & & \downarrow \phi_5 & & \downarrow \phi_6 \\
0 & \longleftarrow & H^6(P(V);\underline{\pi}_6(S^5)) & \longleftarrow & \mathrm{Hom}_{\mathbb{Z}}(C_6,\mathbb{Z}_2) & \xleftarrow{\delta} & \mathrm{Hom}_{\mathbb{Z}}(C_5,\mathbb{Z}_2)
\end{array}$$

$$\text{where } \begin{cases} \delta : \text{coboundary maps} \\ \phi, \phi_5, \phi_6 : \text{forgetfull maps} \end{cases}$$

- C_5 & C_6 are free $\mathbb{Z}[\mathbb{Z}_2]$-modules $\Rightarrow$ ϕ is $(\times 2)$-map.
- Note $H^6(P(V);\pi_6(S^5)) = \mathbb{Z}_2$.

$$\Rightarrow \boxed{\phi \text{ is 0-map}}$$