The Seiberg-Witten equations for families and diffeomorphisms of 4-manifolds

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Introduction
The Seiberg-Witten equations on a family
Applications to diffeomorphisms

Reference

The Seiberg-Witten equations for families and diffeomorphisms of 4-manifolds,
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Introduction

The Seiberg-Witten gauge theory

- X: a smooth closed oriented 4-manifold.
  c: a Spin^c-structure on X.
  $\rightarrow$ The Seiberg-Witten equations on c.

- The moduli space

$$\mathcal{M}_c = \{ \text{ solutions to SW eqns } \}/\mathcal{G},$$

  where $\mathcal{G}$ is the gauge transformation group.

Properties of the moduli

- $\mathcal{M}_c$ is a compact space.
- A generic choice of parameters (metric & perturbation)
  $\to \mathcal{M}_c$ becomes a finite dimensional manifold except
  quotient singularities = reducibles.
- If no reducible, the Seiberg-Witten invariant can be defined from the fundamental class of $\mathcal{M}_c$.

But in some cases,

reducibles play special roles.

The role of reducibles

The case when $b^+ = 0$

Donaldson's "Theorem A" can be proved by using the SW equations.

Theorem A (Donaldson)

X: definite $\Rightarrow$ its intersection form $\cong$ diagonal.

The idea of the proof by SW.

For simplicity, suppose $b_1 = 0$

If negative definite $\Rightarrow \mathcal{M}_c$ always contains exactly one reducible.

$\Rightarrow \dim \mathcal{M}_c \leq 0$.

$\Rightarrow$ A constraint on characteristic elements.

$\underset{\text{[Elkies]}}{\Rightarrow}$ Diagonal. □

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The role of reducibles

The case when $b^+ = 1$

If $b^+ \geq 1 \Rightarrow$ Can avoid reducibles by choosing generic parameters.

$\longrightarrow$ Seiberg-Witten invariants

But, if $b^+ = 1 \Rightarrow$ SW invariants depend on parameters.

The phenomenon called "Wall crossing" occurs.

- The wall crossing is used to prove the Thom conjecture by Kronheimer-Mrowka.

- $b^+ = 0 \Rightarrow$ 0-dimensional family contains a reducible.
  $\Rightarrow$ Theorem A.
- $b^+ = 1 \Rightarrow$ 1-dimensional family may contain a reducible.
  $\Rightarrow$ Wall crossing.
  ...
- $b^+ = k \Rightarrow k$-dimensional family may contain a reducible.

$\Rightarrow$ What can we conclude?

Consider d-dimensional family of a 4-manifold X:

$$\begin{array}{l} \mathbb{X} \\ \downarrow X \\ B \end{array} \quad \leftarrow \text{a fiber bundle over a } d\text{-dim. mfd } B \text{ with fiber } X$$

- If $d < b^+(X) \Rightarrow$ can avoid reducibles.
  $(\because)$The wall has codimension$=b^+(X)$ in the parameter space.
- If $d \geq b^+(X) \Rightarrow$ can not avoid reducibles in general.
  $\rightarrow$ We concentrate on the case when $d = b^+(X)$.
- An argument analogous to the wall crossing
  $\rightarrow$ Constraints on the topology of $\mathbb{X}$.

The case when $d = b^+ = 1$

- X: smooth, closed, oriented, $b_1 = 0$, $b^+ = 1$.
- c: Spin^c-structure s.t. $d(c) := \dim \mathcal{M}_c = 0$.
- $f : X \to X$ an ori. pres. diffeo preserving c.

Consider the mapping torus

$$\begin{array}{l} X_f := X \times [1,0]/f \\ \downarrow \\ S^1. \end{array}$$

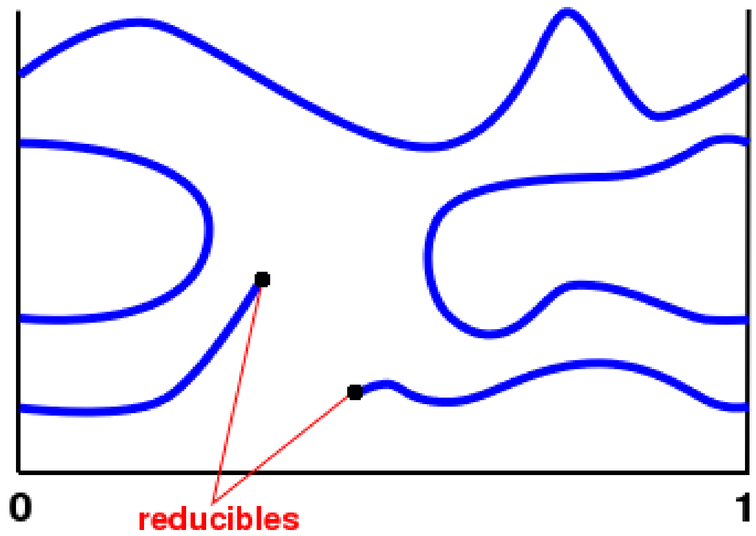
- c induces a Spin^c-str. $\tilde{c}$ on X_f.

$\to$ The SW eqns on $(X_f, \tilde{c})$.

A generic choice of parameters

$\to \mathcal{M}_c$ is a compact 1-dim. manifold with boundaries.

Boundaries are reducibles.



- Note that #reducibles is even.
- On the other hand, #reducibles is related to the action of f on $H^+(X;\mathbb{R})$.

$$\begin{array}{l} H_f^+ := (H^+(X;\mathbb{R}) \times [0,1])/f \\ \quad\downarrow \\ \quad S^1 \end{array}$$

  - Take a section $s\colon S^1 \to H_f^+$ s.t. $s \pitchfork$ (0-section).
  - We will see

$$\#\text{reducibles} \equiv s^{-1}(0) \mod 2.$$

$\Rightarrow H_f^+$ is a trivial bundle.
$\Rightarrow f$ preserves the orientation of $H^+(X;\mathbb{R})$.

Summary

- X: $b_1 = 0$ & $b^+ = 1$.
- c: Spin^c-str. s.t. $\dim \mathcal{M}_c = 0$.
- $f: X \to X$ ori. pres. diffeo. preserving c.

$\Rightarrow f$ preserves the orientation of $H^+(X;\mathbb{R})$.

More concretely,

Theorem

- X: $b_1 = 0$ & *The intersection form* $\cong E_8 \oplus H$.
- $f : X \to X$ *ori. pres. diffeo. s.t.*
  1. f *preserves a class* $C \in \operatorname{Tor} H^2(X; \mathbb{Z})$ *s.t.* $C \to w_2(X)$ *mod* 2,
  2. $H^1(X; \mathbb{Z}_2)^{f^*} = 0$.

$\Rightarrow$ *f preserves the orientation of* $H^+(X; \mathbb{R})$.

Remarks

- The conditions 1. 2. $\Rightarrow f$ preserves the Spin^c-str of C.
- This could be proved by an ordinary wall crossing argument.
- *Cf.* [Donaldson]
  If an ori. pres diffeo of $X = K3$ acts trivially on $H^2(X; \mathbb{Z}_2)$
  $\Rightarrow$ it preserves the ori. of $H^+(X; \mathbb{R})$.

The case when $d = b^+ = 2$

Let $B = T^2 \Rightarrow$ a constraint on commutative two diffeos.

(The precise statement will be given later.)

The Seiberg-Witten equations on a family

- X: an oriented closed 4-manifold, $b_1 = 0$.
- B: an oriented closed d-manifold.

$$\begin{array}{l} \mathbb{X} \\ \downarrow X \\ B \end{array} \quad \leftarrow \text{a fiber bundle over } B \text{ with fiber } X$$

$$T(\mathbb{X}/B) = \coprod_{b \in B} TX_b \quad \leftarrow \text{the tangent bundle along fiber}$$

$$\downarrow \mathbb{R}^4$$

$$\mathbb{X} = \coprod_{b \in B} X_b$$

- Choose a metric on $T(\mathbb{X}/B)$. Then,

$$\begin{array}{l} Fr \longleftarrow \text{the frame bundle} \\ \downarrow \mathrm{SO}(4) \\ \mathbb{X}. \end{array}$$

- A Spin^c-structure on $T(\mathbb{X}/B)$ is a lift of Fr to a $\mathrm{Spin}^c(4)$-bundle $\mathbb{P}$:

$$\begin{array}{ccc} Fr & \longleftarrow & \mathbb{P} \\ \downarrow \mathrm{SO}(4) & & \downarrow \mathrm{Spin}^c(4) \\ \mathbb{X} & = & \mathbb{X}. \end{array}$$

- Suppose a Spin^c-str. on $T(\mathbb{X}/B)$ is given.
  $\Rightarrow$ By restriction, we have a Spin^c-str. on each fiber X_b.

$$\begin{array}{ccc} \mathbb{P} & & P_b := \mathbb{P}|_{X_b} \\ \downarrow & \Rightarrow & \downarrow \\ \mathbb{X} & & X_b \end{array}$$

- Conversely, suppose a Spin^c-str. c on X is given.
  Want to construct a Spin^c-str. on $T(\mathbb{X}/B)$ from c.

In general $\Rightarrow$ Not possible.
Under some conditions $\Rightarrow$ Possible.

Proposition

Suppose $C \in H^2(X;\mathbb{Z})$ and $\mathbb{X} \to B$ satisfy the following

1. $C \mapsto w_2(X)$ *and* $C \in H^2(X;\mathbb{Z})^{\pi_1(B)}$.
2. $H^3(B;\mathbb{Z}) = 0$
3. *The mod 2 reduction* $H^2(B;\mathbb{Z}) \to H^2(B;\mathbb{Z}_2)$ *is surjective.*
4. $H^1(B;\mathcal{H}) = 0$, *where $\mathcal{H}$ is the Serre local system of* $H^1(X;\mathbb{Z}_2)$.

$\Rightarrow \exists$ *a* Spin^c*-str on* $T(\mathbb{X}/B)$ *s.t.* $c_1(\det \mathbb{P}|_{X_b}) = C$.

The idea of proof

To Do = To construct an integral lift $\tilde{c}$ of $w_2(T(\mathbb{X}/B))$.

- If $b_1 = 0$, $E_2^{p,1} = 0, \forall p$ in $\mathbb{Z}$-coefficient, then

$$0 \to H^2(B;\mathbb{Z}) \to H^2(\mathbb{X};\mathbb{Z}) \to H^2(X;\mathbb{Z})^{\pi_1(B)} \to H^3(B;\mathbb{Z}).$$

- By the assumption (4) $H^1(B;\mathcal{H}) = E_2^{1,1} = 0$ in $\mathbb{Z}_2$-coefficient, we have,

$$H^1(X;\mathbb{Z}_2)^{\pi_1(B)} \to H^2(B;\mathbb{Z}_2) \to H^2(\mathbb{X};\mathbb{Z}_2) \to \mathrm{Ker}\, d_3.$$

Compare these exact sequences.

The Seiberg-Witten equations on a family

- Fix a class $C \in H^2(X;\mathbb{Z})$ as in Proposition,
  & fix a Spin^c-str. $\tilde{c}$ on $T(\mathbb{X}/B)$ associated to C.

  $S^\pm$: posi./nega. spinor bundles. L : the determinant line bundle.

  $$S^\pm \to \mathbb{X} \qquad L \to \mathbb{X}$$

- L is viewed as a family over B: $$L = \coprod_{b\in B} L_b \to B$$
- $\mathcal{A}(L_b) := \{\mathrm{U}(1) - \text{connections on } L_b\}$.

- The bundle of parameters:

  $$\Pi := \{(g_b, \mu_b) \in Met(X_b) \times \Omega^2(X_b) \,|\, *_{g_b} \mu_b = \mu_b\} \to B$$

- Choose a section $\eta\colon B \to \Pi$.

The Seiberg-Witten equations for $\{(A_b, \psi_b)\} \in \coprod \mathcal{A}(L_b) \times \Gamma(S_b^+)$.

$$(SW_b) \begin{cases} D_{A_b}\psi = 0, \\ F_{A_b}^{+g_b} = (\psi_b \otimes \psi_b^*)_0 + i\mu_b, \end{cases}$$

where

- $D_{A_b} \colon \Gamma(S_b^+) \to \Gamma(S_b^-)$, the Dirac operator,
- $F_{A_b}^{+g_b}$: the g_b-self-dual part of the curvature of A_b,
- $(\psi_b \otimes \psi_b^*)_0$: the trace free part of $\psi_b \otimes \psi_b^* \in \Gamma(End(S_b^+))$ identified with i-valued 2-form via the Clifford multiplication.

- The gauge transformation group

$$\mathcal{G}_b := \mathrm{Map}(X_b, S^1) \curvearrowright \mathcal{A}(L_b) \times \Gamma(S_b^+),$$
$$u_b(A_b, \psi_b) = (A_b - 2u_b^{-1}du_b, u_b\psi_b).$$

- (SW_b) is $\mathcal{G}_b$-equivariant.
- $\mathcal{G}_b$-action is free where $\psi \not\equiv 0$.
  $\rightarrow$ A solution with $\psi \equiv 0$ is called a *reducible*.
  The stabilizer $\cong S^1$

- Note $(A_b, 0)$ reducible $\Leftrightarrow$ $F^{+g_b}_{A_b} = i\mu_b$.
- The Wall $\mathcal{W}$,

$$\begin{aligned}\mathcal{W} :=& \{(g_b, \mu_b) \,|\, (SW_b) \text{ has a reducible solution } \} \\ =& \{(g_b, \mu_b) \,|\, P_{+g_b}(2\pi C - \mu_b) = 0\},\end{aligned}$$

  where P_{+g_b} is the orthogonal projection to g_b-self-dual harmonic part.
- $\Pi \supset \mathcal{W}$: codimension $= b^+(X)$.

- The moduli space for the family,

$$\mathcal{M}_\eta(\mathbb{X}, \mathbb{P}) = \coprod_{b \in B} \{ \text{ solutions to } (SW_b) \, \} / \mathcal{G}_b.$$

- The virtual dimension of $\mathcal{M}_\eta(\mathbb{X}, \mathbb{P})$:

$$\begin{aligned} d(C) =& 2 \operatorname{ind}_{\mathbb{C}} D_A - (1 + b^+) + d \\ =& \frac{1}{4}(C^2 - \mathrm{Sign}(X)) - 1 - b_+ + d. \end{aligned}$$

- If η: generic,
  $\Rightarrow \mathcal{M}_\eta(\mathbb{X}, \mathbb{P}) \setminus \{\text{reducibles}\}$ becomes a $d(C)$-dim. manifold.

- Recall the situation:

$$\begin{array}{l} \Pi \ \supset \mathcal{W} \leftarrow \text{codimension}=b^+(X) \\ \eta\!\uparrow\!\downarrow \\ B^d \end{array}$$

- $b_1 = 0$.
  $\Rightarrow$ When η intersects $\mathcal{W}$, a reducible appears in $\mathcal{M}_\eta(\mathbb{X}, \mathbb{P})$.

$$\boxed{\eta \cap \mathcal{W} \stackrel{1:1}{\longleftrightarrow} \{\text{reducibles}\} \subset \mathcal{M}_\eta(\mathbb{X}, \mathbb{P})}$$

- To see $\eta \cap \mathcal{W}$, introduce

$$\begin{array}{c} H^+_\eta \subset \Omega^2(\mathbb{X}/B) \\ \downarrow \\ B, \end{array}$$

$$\text{(The fiber over } b \in B) = \{g_b\text{-self-dual harmonic 2-forms}\} = H^{+g_b}(X_b).$$

- Define the section $s_\eta \colon B \to H^+_\eta$ by

$$s_\eta(b) := P_{+g_b}(2\pi C - \mu_b).$$

$$\boxed{\eta \cap \mathcal{W} \stackrel{1:1}{\longleftrightarrow} s_\eta^{-1}(0)}$$

Now suppose

- $\boxed{d = b^+}$, $\Rightarrow d(C) = 2\,\mathrm{ind}_{\mathbb{C}} D_A - 1 \leftarrow$ odd.
- $\boxed{\eta \pitchfork \mathcal{W}}$, $\Leftrightarrow s_\eta \pitchfork$ (0-section)
- $\boxed{\eta\text{: generic}}$. $\Rightarrow \mathcal{M}_\eta(\mathbb{X}, \mathbb{P}) \setminus \{\text{reducibles}\}$ is a $d(C)$-manifold.

Note

$$\boxed{\eta \cap \mathcal{W} \overset{1:1}{\longleftrightarrow} s_\eta^{-1}(0) \overset{1:1}{\longleftrightarrow} \{\text{reducibles}\}}$$

Theorem

If $d(C) > 0$ & $d(C) \equiv 1$ mod 4,

$$\Rightarrow \boxed{\#s_\eta^{-1}(0) = \#\{\textit{reducibles}\} \textit{ is even.}}$$

For the proof, we need to analyse the situation around a reducible,
$\rightarrow$use the Kuranishi model.

The Kuranishi model

- Suppose $x = (A_0, 0)$ is a reducible solution over $b_0 \in B$.
- For simplicity, suppose $g_b = g_{b_0}$ on a small nbd $U \subset B$ of b_0.
- The slice of $\mathcal{G}$-action at $x = (A_0, 0)$ is given by

$$T_x := \{(b, a, \phi) \in U \times i \ker d^* \times \Gamma(S^+) | \, \|a\| \& \|\phi\| : \text{ small}\}.$$

- Let us consider the following S^1-equiv. map:

$$\begin{aligned}\Psi\colon T_x \to & i\Omega^+(X)\times\Gamma(S^-),\\ (b,a,\phi)\mapsto & (F^+_{A_0+a}-(\phi\otimes\phi^*)_0+i\mu_b, D_{A_0+a}\phi).\end{aligned}$$

- Then
$$\Psi^{-1}(0)/S^1\cong(\text{a nbd. of } x)\subset\mathcal{M}(\mathbb{X},\mathbb{P}).$$
- Ψ is a non-linear Fredholm map.
$\Rightarrow$ The differential $D\Psi$ at 0 is written as
$$(D\Psi)_0\colon F\oplus V\to G\oplus W,$$
where F, G are finite dimensinal subspaces, and
$L:=(D\Psi)_0|_V\colon V\to W$ is a linear isomorphism.

- Let p_G, p_W be orthogonal projections from $G\oplus W$ to G, W.
$\Rightarrow$ $p_W\circ(D\Psi)_0$ is surjective.
$\Rightarrow$ $\exists f$: a diffeo from a nbd of 0 to another nbd s.t.
$$p_W\circ\Psi\circ f=p_W\circ(D\Psi)_0,$$
by the implicit function theorem.
- Finally, Ψ can be identified locally with a map
$$\begin{aligned}\Psi'\colon F\times V \to & G\times W,\\ (u,v)\mapsto & (\alpha(u,v),L(v)),\end{aligned}$$
where L is a linear isomorphism, and $(D\alpha)_0=0$.
- Then $f\colon F\to G, f(u)=\alpha(u,0)$ gives a finite dim model for Ψ s.t. $\Psi^{-1}(0)\cong f^{-1}(0)$ locally. (The Kuranishi model)

- In our situation, Ψ is S^1-equivariant, & all of constructions above can be carried out S^1-equivariantly.

$$\Rightarrow f^{-1}(0)/S^1 \cong \Psi^{-1}(0)/S^1 \cong (\text{a nbd of } x) \subset \mathcal{M}(\mathbb{X}, \mathbb{P}).$$

- Then, explicitly, f could be:

$$f : \mathbb{R}^{b^+} \times \operatorname{Ker} D_{A_0} \to \operatorname{Coker} D_{A_0} \times H^+,$$

where S^1 acts on $\operatorname{Ker} D_{A_0}$ & $\operatorname{Coker} D_{A_0}$ by multiplication, on $\mathbb{R}^{b^+}$ & H^+ trivially.

- $\eta \pitchfork \mathcal{W} \Rightarrow \mathbb{R}^{b^+} \times \{0\}$ is isomorphically mapped to $\{0\} \times H^+$.
- If necessary, perturb the Dirac equation,

$$\Rightarrow \boxed{(\text{a nbd of } x) \cong f^{-1}(0)/S^1 \cong \mathbb{C}^k/S^1 \cong c\,\mathbb{C}\mathrm{P}^{k-1},}$$

where $k = \operatorname{ind}_{\mathbb{C}} D_{A_0}$.

Now, let us prove,

Theorem

If $d(C) > 0$ & $d(C) = 2k - 1 \equiv 1 \mod 4$,

$$\Rightarrow \# s_\eta^{-1}(0) = \#\{reducibles\} \text{ is even.}$$

Proof

- $\mathcal{M}(\mathbb{X}, \mathbb{P}) \setminus \{\text{reducibles}\}$ is a $d(C)$-dim. manifold.
- The nbd of each reducible $\cong c\,\mathbb{C}\mathrm{P}^{(d(C)-1)/2}$.
- Remove cones from $\mathcal{M}_\eta(\mathbb{X}, \mathbb{P})$.

$$\Rightarrow \text{(boundary)} = \coprod \mathbb{C}\mathrm{P}^{\frac{d(C)-1}{2}}.$$

- In the unoriented cobordism group Ω^{4n}, $\mathbb{C}P^{2n}$ is non-trivial.

$\to$ When $\frac{d(C)-1}{2} = 2n$, $\Leftrightarrow d(C) \equiv 1 \bmod 4$

$$\begin{aligned}&\#(\text{components of boundary}) \text{ is even.}\\ &\quad \| \\ &\#(\text{reducibles}) \\ &\quad \| \\ &\# s_\eta^{-1}(0)\end{aligned}$$

$\square$

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Moduli spaces for a family

Remark 1

If $H^+_\eta \to B$ is orientable $\Rightarrow \mathcal{M}_\eta(\mathbb{X}, \mathbb{P})$ is orientable.
$\Rightarrow$ Can refine Theorem:

If $d(C) > 0$ & $d(C) \equiv 1 \bmod 4 \Rightarrow \# s_\eta^{-1}(0) = 0$.

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Remark 2

When $d(C) > 0$ & $d(C) \equiv 3 \bmod 4$, Theorem holds in some cases.

$$\mathcal{B}_b^* := \left[\mathcal{A}(L_b) \times (\Gamma(S_b^+) \setminus \{\psi_b \equiv 0\})\right] / \mathcal{G}_b.$$

- If $b_1 = 0 \Rightarrow \mathcal{B}_b^* \simeq \mathbb{C}\mathrm{P}^\infty \Rightarrow \exists U \in H^2(\mathcal{B}_b; \mathbb{Z})$.
- If U has a lift $\tilde{U} \in H^2(\coprod \mathcal{B}_b^*) \Rightarrow$ Theorem holds.
  (∵) A component of the boundary is $\mathbb{C}\mathrm{P}^{2k+1}$.
  Evaluate $[\mathbb{C}\mathrm{P}^{2k+1}]$ by $\tilde{U}^{2k+1}$. □

Applications to diffeomorphisms

- The case when $d = b^+ = 1$.
- The case when $d = b^+ = 2$.

The case when $d = b^+ = 1$

Theorem

- X: *closed, oriented,* $b_1 = 0$.
- *The intersection form* $\Psi_X \cong E_8 \oplus H$.
  $\Rightarrow \exists C \in \mathrm{Tor}\, H^2(X;\mathbb{Z})$ *s.t.* $C \mapsto w_2(X)$ *mod* 2.
- $f : X \to X$ *ori. pres. diffeo. s.t.*
  1. f *preserves a class* $C \in \mathrm{Tor}\, H^2(X;\mathbb{Z})$ *s.t.* $C \to w_2(X)$ *mod* 2,
  2. $H^1(X;\mathbb{Z}_2)^{f^*} = 0$.

$\Rightarrow$ *f preserves the orientation of* $H^+(X;\mathbb{R})$.

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Proof

- Consider the mapping torus:

$$\mathbb{X} := (X \times [0,1])/f \to S^1$$

- By 1 and 2, $\exists\,\mathrm{Spin}^c$-str. $\tilde{c}$ on $\mathbb{X}$ s.t. $c_1(\det \mathbb{P}|_X) = C$.
  $\Rightarrow$ The SW-moduli for $\tilde{c}$.

$$d(C) = \frac{1}{4}(C^2 - \mathrm{Sign}(X)) - 1 = \frac{1}{4}(0 - (-8)) - 1 = 1.$$

- By Theorem, $\# s_\eta^{-1}(0) \equiv 0 \bmod 2$.
- If f reverse the ori. of H^+
  $\Rightarrow H^+_\eta \to S^1$ is a nontrivial $\mathbb{R}$-bundle. $\Rightarrow$ A Contradiction. $\square$

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Remark

If X= Enriques surface, $\Rightarrow \Psi_X \cong E_8 \oplus H$.
But No diffeo can satisfy 2. $H^1(X;\mathbb{Z}_2)^{f^*} = 0$. $(\because) H^1(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Corollary

X: Enriques,
N: a rational homology 4-sphere s.t. $H^1(N;\mathbb{Z}_2) \neq 0$.
$X' = X\#N$.
$\Rightarrow$ No diffeo $f : X' \to X'$ satisfying the following at the same time:

1. *f preserves a class $C \in \mathrm{Tor}\, H^2(X;\mathbb{Z})$ s.t. $C \mapsto w_2(X)$ mod 2,*
2. *$H^1(X;\mathbb{Z}_2)^{f^*} = 0$.*
3. *f reverses the orientation of $H^+(X;\mathbb{R})$.*

The case when $d = b^+ = 2$

Theorem

X: $b_1 = 0$ & $\Psi_X \cong E_8 \oplus H \oplus H$.
$f, g : X \to X$, ori. pres. diffeos s.t. $f \circ g = g \circ f$.
$\Rightarrow$ The following can not hold at the same time:

1. *f and g preserve a class $C \in \mathrm{Tor}\, H^2(X;\mathbb{Z})$ s.t. $C \mapsto w_2(X)$.*
2. *$f^*, g^* : H^1(X;\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_2)$,*
$$\mathrm{rank}\begin{pmatrix} Id - g* \\ -(id - f^*) \end{pmatrix} = \dim H^1(X;\mathbb{Z}_2).$$
3. *$\exists$ positive definite subspace $H^+ \subset H^2(X;\mathbb{R})$ which decomposes $H^+ \cong \mathbb{R} \oplus \mathbb{R}$ on which*
$$f^* = (-1) \oplus (+1),$$
$$g^* = (+1) \oplus (-1).$$

Proof

- Let us consider the "double" mapping torus:

$$\begin{array}{l} \mathbb{X} := (X \times [0,1] \times [0,1])/f, g \\ \downarrow \\ T^2. \end{array}$$

- If 1. & 2., $\Rightarrow \exists\ \mathrm{Spin}^c$-str. $\tilde{c}$ on $\mathbb{X}$.
  $\Rightarrow$ The SW-moduli for $\tilde{c}$.
- $d(C) = 1 \therefore \# s_\eta^{-1}(0) \equiv 0 \bmod 2.$

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The case when $d = b^+ = 2$

- If 3. holds, then

$$\begin{array}{ccc} H_\eta^+ & = & \pi_1^* E \oplus \pi_2^* E \\ \downarrow & & \downarrow \\ T^2 & = & S^1 \times S^1, \end{array}$$

where
  - $\pi_i \colon S^1 \times S^1 \to S^1$, the i-th projection,
  - $E \to S^1$, a nontrivial $\mathbb{R}$-bundle.

$\therefore$ $\boxed{w_2(H_\eta^+) \neq 0.}$ $\to$ Contradicts with $\# s_\eta^{-1}(0) \equiv 0 \bmod 2.$ $\square$

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Remark

In this case, the fiberwise dimension of the moduli is -1.

Question

How about the following cases?

- $d = b^+ = 2$ & $B \neq T^2$.
- $d = b^+ \geq 3$.
- $d > b^+$.
- $b_1 > 0$.