Bauer-Furuta invariants and a non-smoothable involution on $K3\#K3$

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Main Theorem

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There exists a locally linear $\mathbb{Z}_2$-action on $X = K3\#K3$ which can not be smooth w.r.t. any smooth structure on X.

Introduction

Theorem (Liu-N. '05-06)

There exist loc. lin. $\mathbb{Z}_p$-actions ($p = 3, 5, 7$) on $K3$ which can not be smooth w.r.t. infinitely many smooth structures including the standard.

The proof consists of 2-steps:

1. Existence: To construct loc. lin. actions concretely
   $\rightarrow$ Edmonds-Ewing's realization theorem of loc. lin. actions.
2. Non-smoothable: To prove actions in 1. do not satisfy the conditions to be smooth.
   $\rightarrow$ Seiberg-Witten gauge theory

Non-smoothability

Gauge theory $\longrightarrow$ Constraints on smooth actions
$\longrightarrow$ Non-smoothable

- Mod p vanishing theorem [Fang] ([N.])
  Some conditions on fixed point data $\Rightarrow \mathrm{SW}_X(c) \equiv 0 \bmod p$.
- $\mathrm{SW}_{K3}(c_0) = 1$ for the spin structure c_0.
  $\Rightarrow$ Not "(some conditions)".

- But we can not use this method when $\mathrm{SW}_X \equiv 0$.
- Bauer and Furuta defined a stable cohomotopy refinement of SW-invariants. $\longrightarrow$ Bauer-Furuta invariants
  e.g. $X = K3\#K3 \Rightarrow \mathrm{SW}_X \equiv 0$ but $\mathrm{BF}_X \not\equiv 0$.

Question

1. Does "mod p vanishing theorem" for BF-inv. hold?
2. Can we construct a non-smoothable action on $K3\#K3$?

$\longrightarrow$ Yes for both 1. and 2.

1. A vanishing theorem of BF-inv. under involutions
2. Main Theorem

As a byproduct, we also have;

Theorem

There exists a loc. lin. $\mathbb{Z}_2$-action on $X = K3$ s.t.

1. $X^{\mathbb{Z}_2}$: *discrete*
2. $b_+^{\mathbb{Z}_2} := \dim H^+(X;\mathbb{R})^{\mathbb{Z}_2} = 3$.
3. *nonsmoothable for any smooth structure.*

Cf. [Bryan, '98] For smooth $\mathbb{Z}_2$-actions on $K3$,

- $X^{\mathbb{Z}_2}$: discrete & $b_+^{\mathbb{Z}_2} = 3$,
  or
- $\dim X^{\mathbb{Z}_2} = 2$ & $b_+^{\mathbb{Z}_2} = 1$.

Preliminaries

Definition

G: a finite group, X: a n-dim. C^0-manifold.
A topological G-action on X is called locally linear if $\forall x \in X$, $\exists V_x$: a G_x-invariant nbd. of x, (G_x: isotropy group of x.) s.t.

- $V_x \cong \mathbb{R}^n$,
- G_x acts on $\mathbb{R}^n$ in linear orthogonal way.

In general,

$$\text{smooth} \quad \Rightarrow \quad \text{locally linear}$$
$$\not\Leftarrow$$

Non-smoothable G-actions

X: a C^0-manifold,
$X_\sigma \leftarrow$ a smooth structure σ specified.

$$LL(G,X) := \{\text{loc. lin. } G\text{-actions on } X\}/ \sim_{homeo},$$
$$C^\infty(G,X_\sigma) := \{\text{smooth } G\text{-actions on } X_\sigma\}/ \sim_{diffeo}.$$

$\varphi_\sigma\colon C^\infty(G,X_\sigma) \to LL(G,X)$ $\rightarrow$forgetting the smooth structure

Definition

A loc. lin. G-action on X is

- non-smoothable w.r.t. σ if (Its class) $\notin \mathrm{im}\,\varphi_\sigma$,
- smoothable w.r.t. σ if (Its class) $\in \mathrm{im}\,\varphi_\sigma$.

Fact

If $n = \dim X \leq 3 \Rightarrow$ No non-smoothable loc. lin. action.

$n = 4$

$\exists$ Many examples of non-smoothable actions.

1. [Kwasik-Lee '88] $G = \mathbb{Z}_2 \curvearrowright X$: a closed smooth 4-manifold.
2. [Kwasik-Lawson '93]
   $G = \mathbb{Z}_p$ (p: prime)$\curvearrowright X$: contractible s.t.
   $\partial X = \Sigma(a, b, c)$: Brieskorn.
3. [Hambleton-Lee '95] $G = \mathbb{Z}_5 \curvearrowright X = \mathbb{C}P^2 \# \mathbb{C}P^2$.
4. [Bryan '98] $G = \mathbb{Z}_2 \curvearrowright X = K3$.
5. [Kiyono '04] $G = \mathbb{Z}_p$ (p: prime)$\curvearrowright X = \# S^2 \times S^2$.
6. [Liu-N '05-06] $G = \mathbb{Z}_p$ $(p = 3, 5, 7)\curvearrowright X = E(n)$.
7. [Chen-Kwasik '07] $\exists$family of symplectic exotic $K3$ s.t. $\forall$nontrivial odd order loc. lin. actions are non-smoothable.
8. [N. '07] $G = \mathbb{Z}_2 \curvearrowright X = K3\#K3$.

Construction of a non-smoothable $\mathbb{Z}_2$-action on $K3\#K3$

- A vanishing theorem of Bauer-Furuta invariants under $\mathbb{Z}_2$-actions
- A constraint on smooth $\mathbb{Z}_2$-actions on $K3\#K3$
- Edmonds-Ewing's construction of loc. lin. actions.
- Construction of a non-smoothable $\mathbb{Z}_2$-action

A vanishing theorem of Bauer-Furuta invariants under $\mathbb{Z}_2$-actions

Suppose

- $G = \mathbb{Z}_2$ acts on a smooth closed oriented X^4 smoothly.
- the $\mathbb{Z}_2$-action lifts to a Spin^c-structure c.

Fix a $\mathbb{Z}_2$-invariant metric
and a $\mathbb{Z}_2$-invariant connection A_0 on the determinant line bundle L.
$\rightarrow D_{A_0} : \Gamma(S^+) \rightarrow \Gamma(S^-)$ $\mathbb{Z}_2$-equivariant Dirac operator.
Then,

$$\mathrm{ind}_{\mathbb{Z}_2} D_{A_0} = k_+\mathbb{C}_+ + k_-\mathbb{C}_- \in R(\mathbb{Z}_2) \cong \mathbb{Z}[t]/(t^2 - 1),$$

where

- $\mathbb{Z}_2$ acts on $\mathbb{C}_+$ trivially,
- $\mathbb{Z}_2 = \{\pm 1\}$ acts on $\mathbb{C}_-$ by multiplication.

Theorem (Vanishing theorem of BF)

Suppose

1. $b_1 = 0,\ b_+ \geq 2,\ b_+^{\mathbb{Z}_2} = \dim H^+(X;\mathbb{R})^{\mathbb{Z}_2} \geq 1.$
2. $d(c) := 2(k_+ + k_-) - (1 + b_+) = 1.$
3. $2k_\pm < 1 + b_+^{\mathbb{Z}_2}.$
4. $b_+ - b_+^{\mathbb{Z}_2}$ *is odd.*

Then $\mathrm{BF}(c) = 0$.

Remark

- *$d(c)$ is the virtual dimension of the SW-moduli for c.*
- *When $d(c) = 1$,*
  - *$k_+ + k_-$: even $\Rightarrow$ $BF(c) \in \mathbb{Z}/2$.*
  - *$k_+ + k_-$: odd $\Rightarrow$ $BF(c) = 0$.*
  - *always* $\mathrm{SW}_X(c) = 0$.

A constraint on smooth actions on $K3\#K3$

Suppose

- X: smooth, closed, oriented, spin, $\pi_1(X) = 1$.
- $\mathbb{Z}_2 \curvearrowright X$ smoothly.

If $X^{\mathbb{Z}_2}$: discrete $\Rightarrow$ the $\mathbb{Z}_2$-action lifts to the spin c_0.

G-spin theorem

$$k_+ - k_- = \frac{1}{4} \sum_{p \in X^{\mathbb{Z}_2}} \varepsilon(p),$$

$$k_+ + k_- = -\frac{1}{8} \operatorname{Sign}(X),$$

where $\varepsilon\colon X^{\mathbb{Z}_2} \to \{\pm 1\}$ is a function determined from the lift of the action.

$$\therefore \boxed{2k_\pm = -\frac{1}{8}\operatorname{Sign}(X) \pm \frac{1}{4}\sum \varepsilon(p).}$$

$$\to \sum \varepsilon(p) \equiv 0 \mod 8.$$

Theorem (Furuta-Kametani-Minami)

X: homotopy $K3\#K3 \Rightarrow BF(c_0) \neq 0 \in \mathbb{Z}/2$.

Proposition (Constraints on smooth actions)

- $\mathbb{Z}_2 \curvearrowright X$: *homotopy* $K3\#K3$
- $X^{\mathbb{Z}_2}$: *discrete*
- $b_+^{\mathbb{Z}_2} = 5$ (*Cf.* $b_+ = 6$.)

Then $|\sum \varepsilon(p)| \geq 8$.

Proof.

If $\sum \varepsilon(p) = 0 \Rightarrow 2k_\pm = 4 \pm \frac{1}{4}\sum \varepsilon(p) < 6 = 1 + b_+^{\mathbb{Z}_2}$.
$\therefore BF(c_0) = 0.$ ← A Contradiction. □

Atiyah-Bott's criterion for ε

$\iota\colon X \to X$ involution $\Rightarrow$ involution on the frame bundle $\iota_*\colon F \to F$.
A spin structure is given by $\varphi\colon \hat{F} \overset{2:1}{\to} F$.
For $P, Q \in X^{\mathbb{Z}_2}$, want to compare $\varepsilon(P)$ & $\varepsilon(Q)$.
Take $y \in F_P$, $y' \in F_Q$ and a path s connecting y and y'.
Note $\iota_* y = -y$ and $\iota_* y' = -y'$.

$$C = s \cup -\iota_* s \leftarrow \text{a circle}$$

Proposition

$\varphi^{-1}(C)$ has 2-components $\Leftrightarrow \varepsilon(P) = \varepsilon(Q)$.
($\varphi^{-1}(C)$ is connected $\Leftrightarrow \varepsilon(P) = -\varepsilon(Q)$.)

Edmonds-Ewing's construction of loc. lin. actions

Theorem (Edmonds-Ewing '92)

$\Psi\colon V \times V \to \mathbb{Z}$ a $\mathbb{Z}_2$-inv. symm. unimodular even form s.t.

1. *As a $\mathbb{Z}[\mathbb{Z}_2]$-module, $V \cong T \oplus F$,*

$$\text{where } T \cong n\mathbb{Z} \leftarrow \text{a trivial } \mathbb{Z}[\mathbb{Z}_2]\text{-module}$$
$$F \cong k\mathbb{Z}[\mathbb{Z}_2] \leftarrow \text{a free } \mathbb{Z}[\mathbb{Z}_2]\text{-module}$$

2. *$\forall v \in V, \Psi(gv, v) \equiv 0$ mod 2.*
3. *G-signature formula $\mathrm{Sign}(g, (V, \Psi)) = 0$.*

$\Rightarrow \exists$loc. lin $\mathbb{Z}_2$-action on a simply-connected 4-manifold X s.t.

- *Its intersection form $= \Psi$,*
- *$\# X^{\mathbb{Z}_2} = n + 2$.*

Remark

Since Ψ is supposed even, the homeotype of X is unique

Idea of Proof $\to$ Equivariant handle construction

$$\text{A unit 4-ball } B_0 \subset \mathbb{C}^2 \curvearrowleft \{\pm 1\}$$
$$T \leftrightarrow H_1, \dots, H_n : \text{ copies of } D^2 \times D^2 \subset \mathbb{C}^2 \curvearrowleft \{\pm 1\}$$
$$F \leftrightarrow \text{ free 2-handles}$$

Note: $B_0^{\mathbb{Z}_2} = \{0\}$, $(D^2 \times D^2)^{\mathbb{Z}_2} = \{0\}$.

Step 1.

Represent Ψ by a $\mathbb{Z}_2$-invariant framed link L in ∂B_0.

- By changing basis,

$$\Psi|_T \cong (a_{ij}) \text{ s.t. } \begin{cases} a_{ii} : \text{even} \\ a_{ij} : \text{odd } (i \neq j). \end{cases}$$

- K, K': $\mathbb{Z}_2$-invariant knots in ∂B_0.

$$\Rightarrow lk(K, K') = \text{ odd }.$$

$\longrightarrow$ Can represent $\Psi|_T$ by a framed link L_T.
$\longrightarrow$ Easy for the free part of $\Psi \to L$.

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A vanishing theorem of Bauer-Furuta invariants
A constraint on smooth actions on $K3\#K3$
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Construction of a non-smoothable action

Step2. Attach $H_1, \ldots, H_n$ and free handles to B_0 equivariantly along L.

$$\longrightarrow \mathbb{Z}_2 \curvearrowright X_0 := B_0 \cup H_1 \cup \cdots \cup H_n \cup (\text{free handles}).$$

The $\mathbb{Z}_2$-action on X_0 is smooth.

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Step3. Note

- $\Sigma := \partial X_0$: a $\mathbb{Z}$-homology 3-sphere,
- $\mathbb{Z}_2 \curvearrowright \Sigma$: free.

Theorem ([EE])

Under the above assumptions, $\exists$loc. lin $\mathbb{Z}_2$-action on W^4 s.t.

- *W: contractible,*
- *$(\mathbb{Z}_2 \curvearrowright \partial W) = (\mathbb{Z}_2 \curvearrowright \Sigma)$,*
- *$W^{\mathbb{Z}_2} = \{1 \text{ point}\}$.*

$$\to \boxed{\mathbb{Z}_2{}^{\exists} \curvearrowright X = X_0 \cup_\Sigma W, \text{ locally linear.}}$$

Note the above action is smooth on X_0.
$\to$ Can determine ε on $X_0 = B_0 \cup H_1 \cup \cdots \cup H_n \cup$ (free handles).

- Each of $B_0, H_1, \ldots, H_n$ has one fixed point: $P, Q_1, \ldots, Q_n$.
- Compare $\varepsilon(P)$ with $\varepsilon(Q_i)$, $i = 1, \ldots, n$.

$$\begin{array}{ccccc} L = & K_1 \cup \cdots \cup & K_n & \cup \cdots, \\ & \updownarrow a_{11} & \updownarrow a_{nn} & \\ & H_1 & H_n & \end{array}$$

Proposition

Suppose K_i is a trivial knot.

$$a_{ii} \equiv 2 \mod 4 \Leftrightarrow \varepsilon(P) = \varepsilon(Q_i),$$
$$a_{ii} \equiv 0 \mod 4 \Leftrightarrow \varepsilon(P) = -\varepsilon(Q_i).$$

Construction of a non-smoothable action on $K3\#K3$

$X = K3\#K3 \Rightarrow \Psi_X \cong 4E_8 \oplus 6H$.

Define $\mathbb{Z}_2$-action on $4E_8 \oplus 6H$ as follows:

- $\mathbb{Z}_2 \curvearrowright 2E_8 \oplus 2E_8$: Permutation
- $\mathbb{Z}_2 \curvearrowright H \oplus H$: Permutation
- $\mathbb{Z}_2 \curvearrowright 4H$: Trivial

Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} \leftrightarrow \text{indefinite, even, unimodular} \cong 4H$$

The matrix A can be represented by a link L_T whose each component is a trivial knot.

$\because$ Let $p\colon S^3 \to S^2$ be the Hopf fibration. Put $L_T = p^{-1}(8 \text{ points})$

$\rightarrow$ $\boxed{\exists \text{ a loc. lin. action on } X = K3\#K3}$

Note $\mathbb{Z}_2 \curvearrowright X_0 = B_0 \cup (2\text{-handles})$ is smooth.

Proposition

The smooth action on X_0 can not be extended to X smoothly.

Proof.

If smoothly extended $\Rightarrow |\sum \varepsilon(p)| \geq 8$. $\leftarrow$ Impossible for A. □

More strongly,

Theorem

The above action is non-smoothable w.r.t $\forall$ smooth structures.

Difficulty

ε may depend on smooth structures???

$\rightarrow$ Give a topological definition of ε.

- Consider the topological spin structure on the tangent microbundle.
- Define ε for the action on the top. spin s.t.
  - depends only on classes of loc. lin. actions.
  - coincides with the original in the smooth case.

  Use Atiyah-Bott's criterion for ε as the definition.

Completion of the proof

Proof of the non-smoothability.

- If the above action is smoothable w.r.t. some smooth structure,
  $\Rightarrow |\sum \varepsilon(p)| \geq 8$ by the vanishing theorem.
- But this is impossible for the matrix A. $\square$

Similar method $\Rightarrow$ a nonsmoothable $\mathbb{Z}_2$-action on $K3$.

The proof of the vanishing theorem

- Definition of Bauer-Furuta invariants
- Bauer-Furuta invariants as obstruction classes
- Equivariant BF invariants and equivariant obstructions
- The proof of the vanishing theorem

Bauer-Furuta invariants

X: smooth, closed, oriented, $b_1 = 0$, $b_+ \geq 2$, $b_+^{\mathbb{Z}_2} \geq 1$.
c: a Spin^c-structure
$\mathbb{Z}_2 \curvearrowright (X, c)$ ← smoothly
$S^{\pm}$: posi/nega spinor bundle, $L = \det S^+$.
Then $\mathbb{Z}_2 \curvearrowright S^{\pm}, L$.
Fix a G-inv. metric & G-inv. connection A_0 on L.

$$\mathbb{Z}_2 \times S^1 \curvearrowright \begin{array}{l} \mathcal{C} = \Omega^1(X) \oplus \Gamma(S^+), \\ \mathcal{U} = \Gamma(S^-) \oplus i\Omega^+(X) \oplus \operatorname{im} d^* (\subset \Omega^0(X)) \end{array}$$

where

$$\mathbb{C} \supset S^1 \curvearrowright \Gamma(S^{\pm}) \text{ by multiplication,}$$
$$S^1 \curvearrowright \Omega^\bullet(X) \text{ trivially.}$$

Monopole map

Define $\mu\colon \mathcal{C} = \Omega^1(X) \oplus \Gamma(S^+) \to \mathcal{U} = \Gamma(S^-) \oplus i\Omega^+(X) \oplus \operatorname{im} d^*$ by

$$\mu(a,\phi) = (D_{A_0+ia}\phi, F^+_{A_0+ia} - q(\phi), d^*a),$$
$$\text{where } q(\phi) = (\phi \otimes \phi^*)_0 \in \mathfrak{sl}(S^+) \cong \Omega^+ \otimes \mathbb{C}.$$

Then μ is $\mathbb{Z}_2 \times S^1$-equivariant, non-linear Fredholm, proper.

Decompose $\mu = l + c$, as

$$l(a,\phi) = (D_{A_0}\phi, d^+a, d^*a), \quad c = \mu - l.$$

- l: linear
- c: quadratic, compact.

Finite dimensional approximation

Theorem (Bauer-Furuta)

$\exists W_f \subset \mathcal{U}$: a finite dimensional subspace s.t.

- *$W_f + \operatorname{im} l = \mathcal{U}$.*
- *For each finite dim. subsp. $W \supset W_f$, put $V := l^{-1}(W)$,*

$$\mu \longrightarrow \exists f_W\colon S^V \to S^W \text{ a pointed } \mathbb{Z}_2 \times S^1\text{-equiv. map},$$

S^V, S^W: one-point compactifications of V, W based at infinity.
Roughly, $f_W = (l + p_W c)^+$ for some projection p_W.
and, if $W' = U \oplus W \subset \mathcal{U}$

$$\Rightarrow f_{W'} \sim \mathrm{id}_U \wedge f_W\colon S^{V'} \cong S^{U\oplus V} \to S^{W'} \cong S^{U\oplus W}$$

$\uparrow$

$\mathbb{Z}_2 \times S^1$-homotopic

Equivariant Bauer-Furuta invariants

Definition

$\mathbb{Z}_2$-equivariant Bauer-Furuta invariant:

$$\begin{aligned}\mathrm{BF}^{\mathbb{Z}_2}(c) &:= [f_W] \in \{\, \mathrm{ind}_{\mathbb{Z}_2} D, H^+\}^{\mathbb{Z}_2\times S^1} \\ &:= \operatorname*{colim}_{U\subset W^\perp\subset\mathcal{U}} [S^U \wedge S^V, S^U \wedge S^W]^{\mathbb{Z}_2\times S^1}\end{aligned}$$

Definition

(ordinary) Bauer-Furuta invariant:

$$\begin{aligned}\mathrm{BF}(c) &:= [f_W] \in \{\, \mathrm{ind}\, D, H^+\}^{S^1} \\ &:= \operatorname*{colim}_{U\subset W^\perp\subset\mathcal{U}} [S^U \wedge S^V, S^U \wedge S^W]^{S^1}\end{aligned}$$

Relation

$\phi\colon \{\mathrm{ind}_{\mathbb{Z}_2} D, H^+\}^{\mathbb{Z}_2\times S^1} \to \{\mathrm{ind}\, D, H^+\}^{S^1}$ ←forgetting the $\mathbb{Z}_2$-action

$$\boxed{\mathrm{BF}(c) = \phi(\mathrm{BF}^{\mathbb{Z}_2}(c))}$$

The idea of the proof of the vanishing theorem

- Under the assumptions of theorem, we prove ϕ is 0 map.
  →Using equivariant obstruction theory
- The proof is inspired by Bauer's preprint.

Bauer-Furuta invariants as obstructions

Fact

- If $\operatorname{ind} D > 0 \Rightarrow \{S^V, S^W\}^{S^1} \cong \{S^V/S^1, S^W\}$.
- For sufficiently large V, W,

  $\{S^V/S^1, S^W\} \cong [S^V/S^1, S^W] \leftarrow$ Ordinary cohomotopy group

  $\rightarrow$ Can use ordinary obstruction theory.
- $S^V/S^1 \cong \Sigma^k \mathbb{C}\mathrm{P}^m$
  $\because V = a\mathbb{C} \oplus b\mathbb{R}$, $S^1 \curvearrowright \mathbb{C}$ multiplication, $S^1 \curvearrowright \mathbb{R}$ trivial.

Proposition

$d(c) = 1$, $n := \dim S^V/S^1$, ($\Rightarrow \dim S^W = n - 1$.)

$$H^r(S^V/S^1, *; \pi_r(S^W)) = \begin{cases} 0 & (r \neq n), \\ \mathbb{Z}/2 & (r = n). \end{cases}$$

Theorem (*Cf.* [Hu])

$\exists$ a subgroup $J \subset H^n(S^V/S^1, *; \pi_n(S^W))$,

$$\begin{aligned} [S^V/S^1, S^W] &\cong H^n(S^V/S^1, *; \pi_n(S^W))/J, \\ f &\mapsto d(f, \underline{0}) \leftarrow \textit{difference obstruction} \end{aligned}$$

where $\underline{0}\colon S^V \to \{*\} \subset S^W$ the collapsing map.

Corollary

$$\{S^V, S^W\}^{S^1} \cong [S^V/S^1, S^W] \cong \begin{cases} \mathbb{Z}/2 & \mathrm{ind}\, D: \ \textit{even}, \\ 0 & \mathrm{ind}\, D: \ \textit{odd}. \end{cases}$$

Thus $\mathrm{BF}(c)$ can be written as $\mathrm{BF}(c) = d(f_W, \underline{0})$.

Equivariant obstruction theory and equiv. BF invariants

In some cases, equivariant BF invariants $\mathrm{BF}^{\mathbb{Z}_2}(c)$ can be written in terms of equivariant obstruction classes.

$$\left.\begin{matrix}\text{Ordinary cohomology} \\ H^r(S^V/S^1; \pi_r(S^W))\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Bredon cohomology} \\ H^r_{\mathbb{Z}_2\times S^1}(S^V; \underline{\pi}_r(S^W))\end{matrix}\right.$$

ordinary obstruction class $\leftrightarrow$ equivariant obstruction class

Theorem ([Bredon],[Matumoto]...)

Suppose $H^r_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_r(S^W))=0$ *if* $r\neq n=\dim S^V/S^1$.
Then $\exists$ *a subgroup* $J'\subset H^n_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_n(S^W))$,

$$\{S^V,S^W\}^{\mathbb{Z}_2\times S^1}\cong H^n_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_n(S^W))/J'.$$

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The proof of the vanishing theorem

Lemma

If $2k_\pm<1+b_+^{\mathbb{Z}_2}\Rightarrow C^r_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_n(S^W))=0$ *if* $r\leq n-2$.

Lemma

If $b_+-b_+^{\mathbb{Z}_2}$ *is odd* $\Rightarrow H^{n-1}_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_{n-1}(S^W))=0$.

Cf. If $b_+-b_+^{\mathbb{Z}_2}$ is even $\Rightarrow H^{n-1}_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_{n-1}(S^W))\cong\mathbb{Z}_2$.

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Corollary

Suppose

1. $b_1 = 0,\ b_+ \geq 2,\ b_+^{\mathbb{Z}_2} \geq 1$,
2. $d(c) = 1$,
3. $2k_\pm < 1 + b_+^{\mathbb{Z}_2}$
4. $b_+ - b_+^{\mathbb{Z}_2}$: *odd*,

Then,

- $\{S^V, S^W\}^{\mathbb{Z}_2 \times S^1} \cong H^n_{\mathbb{Z}_2 \times S^1}(S^V, *; \underline{\pi}_n(S^W))/J'$,
- $\mathrm{BF}^{\mathbb{Z}_2}(c) = d(f_W, \underline{0})$.

Cf. $\mathrm{BF}(c) \in \{S^V, S^W\}^{S^1} \cong H^n(S^V/S^1, *; \pi_n(S^W))/J$

Compare the ordinary cohomology and the Bredon cohomology in the top degree:

$$\exists\tilde{\phi} : H^n_{\mathbb{Z}_2 \times S^1}(S^V, *; \underline{\pi}_n(S^W)) \to H^n(S^V/S^1, *; \pi_n(S^W)) \cong \mathbb{Z}/2.$$

Claim $\tilde{\phi}$ is 0-map.

In fact, $\tilde{\phi}$ is $(\times 2)$-map.

Consider the commutative diagram:

$$\begin{array}{ccc} H^n(S^V/S^1,*;\pi_n(S^W)) & \longrightarrow & H^n(S^V/S^1,*;\pi_n(S^W))/J \ni \mathrm{BF}(c) \\ {\scriptstyle \tilde{\phi}=0}\uparrow & & {\scriptstyle \phi}\uparrow \\ H^n_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_n(S^W)) & \longrightarrow & H^n_{\mathbb{Z}_2\times S^1}(S^V,*;\underline{\pi}_n(S^W))/J' \ni \mathrm{BF}^{\mathbb{Z}_2}(c). \end{array}$$

$$\Rightarrow \mathrm{BF}(c) = 0$$

Remarks

- We can give an alternative proof of mod p vanishing theorem in the case when $b_1 = 0$ & $d(c) = 0$.
- Suppose $d(c) = 1$ & a $\mathbb{Z}_p$-action (p: odd prime) given.
  - $H^r_{\mathbb{Z}_p\times S^1}(S^V,*;\underline{\pi}_r(S^W)) = 0$ for low r under some conditions.
  - However ϕ is NOT a 0-map.

  $\rightarrow$Can not expect the vanishing theorem.
- $d(c) \geq 2 \rightarrow$ Not easy to prove the vanishing theorem.
  $\because$ $(n-2)$-th cohomology does not vanish.

$G = \mathbb{Z}_2 \curvearrowright (X, c)$ smoothly.

Vanishing theorem of BF

1. $b_1 = 0,\ b_+ \geq 2,\ b_+^{\mathbb{Z}_2} \geq 1$.
2. $d(c) = 1$.
3. $2k_\pm < 1 + b_+^{\mathbb{Z}_2}$.
4. $b_+ - b_+^{\mathbb{Z}_2}$ is odd.

Then $\mathrm{BF}(c) = 0$.

Mod p vanishing theorem of SW

1. $b_1 = 0,\ b_+ \geq 2,\ b_+^{\mathbb{Z}_2} \geq 1$.
2. $2k_\pm < 1 + b_+^{\mathbb{Z}_2}$.

Then $\mathrm{SW}_X(c) \equiv 0 \bmod 2$.

Geometric meaning of $2k_\pm < 1 + b_+^{\mathbb{Z}_2}$

Let $f_W \colon S^V \to S^W$ a finite dimensional approximation.

$$\to f'_W \colon S^V/S^1 \to S^W,\ \mathbb{Z}_2\text{-equivariant}.$$

In general,

$$(\text{The SW-moduli}) = (f'_W)^{-1}(0).$$

Geometric meaning of $2k_{\pm} < 1 + b_+^{\mathbb{Z}_2}$

In fact,

$$2k_{\pm} < 1 + b_+^{\mathbb{Z}_2} \Leftrightarrow \dim(S^V/S^1)^{\mathbb{Z}_2} < \dim(S^W)^{\mathbb{Z}_2}$$

$\Rightarrow$ Can perturb f'_W equivariantly s.t. $(f'_W)^{-1}(0) \cap (S^V/S^1)^{\mathbb{Z}_2} = \emptyset$.

$$\therefore \mathbb{Z}_2 \curvearrowright (f'_W)^{-1}(0) \text{ free}$$

Geometric meaning of $2k_{\pm} < 1 + b_+^{\mathbb{Z}_2}$

When $d(c) = 0$, by Pontrjagin-Thom construction,

$$\mathrm{SW}_X(c) = \mathrm{BF}(c) = \#(f'_W)^{-1}(0) \equiv 0 \mod 2.$$

$(\because)$ $\mathbb{Z}_2 \curvearrowright (f'_W)^{-1}(0)$ free.

$\rightarrow$ Mod 2 vanishing theorem

Geometric meaning of $2k_{\pm} < 1 + b_{+}^{\mathbb{Z}_2}$

When $d(c) = 1$, $(f'_W)^{-1}(0) = \coprod S^1$.
Very roughly,

$$\mathrm{BF}(c) = \#\{\text{components of } (f'_W)^{-1}(0)\} \mod 2.$$

But

$$\mathbb{Z}_2 \curvearrowright (f'_W)^{-1}(0) \text{ free } \not\Rightarrow \mathrm{BF}(c) = 0.$$

($\because$) $\mathbb{Z}_2$ can act one component freely.

We need an extra condition $b_{+} - b_{+}^{\mathbb{Z}_2}$: odd for the vanishing.