Real structures and $Pin^-(2)$-monopole equations

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$\mathrm{Pin}^-(2) = \mathrm{Pin}(2) = U(1) \cup jU(1) \subset Sp(1) \subset \mathbb{H}$

$U(1)$-monopole eqns

$\mathrm{Pin}^-(2)$-monopole eqns = Seiberg-Witten eqns

twisted along local system

or

a real version of SW.

- SW $(U(1))$ is on Spin^c structure $\leftarrow \mathrm{Spin}^c(4) = \dfrac{\mathrm{Spin}(4) \times U(1)}{\pm 1}$
- $\mathrm{Pin}^-(2)$-monopole Spin^{c-} str $\leftarrow \mathrm{Spin}^{c-}(4) = \dfrac{\mathrm{Spin}(4) \times \mathrm{Pin}^-(2)}{\pm 1}$

SW (U(1))

- Intersection form
  - Definite
  - $\frac{10}{8}$-inequality [Furuta]
- SW invariants $SW^{U(1)}$
- Symplectic [Taubes]
  - nonvanishing
  - SW = Gr
- Kähler [Friedman-Morgan] [Okonek-Teleman] [Brussee]...
  Kobayashi-Hitchin type correspondence

$Pin^-(2)$

- Intersection form with local coefficient
- $Pin^-(2)$-monopole invariants SW^{Pin}
- Real symplectic
  - nonvanishing
- Real Kähler
  K-H

Today's topics

Ordinary SW

[Taubes] (X^4, ω) closed symplectic K: canonical line bundle

- $\mathfrak{s}_0$: canonical Spin^c structure

$$SW^{U(1)}(X, \underset{\mathfrak{s}_0 \otimes K}{\mathfrak{s}_0}) = \pm 1$$

- If $SW^{U(1)}(X, \mathfrak{s}) \neq 0$

  $\Rightarrow$ • $|c_1(\det \mathfrak{s})[\omega]| \leq c_1(K)[\omega]$

  • dim. of the moduli $= 0$

- $SW = Gr$

Kähler

K-H type correspondence

Complex line bundle

The moduli space of SW solutions for $s_0 \otimes E$

$$\cong \left\{ (\delta, \alpha) \;\middle|\; \begin{array}{l} \delta : \text{holomorphic str. on } E \quad K\otimes E^{-1} \\ \alpha : \text{nonzero holo. section of } E \quad K\otimes E^{-1} \end{array} \right\} \Big/ e^{\mathbb{C}}$$

if $(2c_1(E) - c_1(K))[\omega] < 0$

> 0

$$\cong \left\{ \begin{array}{l} \text{effective divisors whose line bundle} \\ \quad D \qquad\qquad \mathcal{L}_D \underset{C^\infty}{\cong} E \quad K\otimes E^{-1} \end{array} \right\}$$

Our setting

(X^4, ω) : symplectic — real symplectic 4-mfd <u>without real part</u>

$\iota \curvearrowright$ $\boxed{\text{free}}$ involution s.t. $\boxed{\iota^*\omega = -\omega}$ $\Longleftrightarrow$ $\iota_* \circ J = -J \circ \iota_*$

J: compatible almost cpx str.

$$\begin{array}{c} X \\ \pi \downarrow 2:1 \\ \hat{X} := X/_{\iota} \end{array} \qquad \begin{array}{c} \ell := X \times_{\{\pm 1\}} \mathbb{Z} \\ \downarrow \\ \hat{X} \end{array} \qquad \ell_{\mathbb{R}} := \ell \otimes \mathbb{R}$$

$\exists \hat{\omega} \in \Omega^2(\hat{X}; \ell_{\mathbb{R}})$ s.t. $\pi^*\hat{\omega} = \omega$

"twisted" symplectic form

anti linear lift, $I^2 = 1$

$$\exists I \curvearrowright \begin{array}{c} K \\ \downarrow \mathbb{C} \\ \iota \curvearrowright X \end{array} \text{ canonical line b'dl} \qquad \begin{array}{c} \hat{K} := K/_I \\ \downarrow \mathbb{R}^2 \\ \hat{X} \end{array} \text{ s.t.} \quad \begin{array}{l} \cdot\ \pi^*\hat{K} = K \\ \cdot\ \det \hat{K} = \ell_{\mathbb{R}} \end{array}$$

$\boxed{\text{Thm}}$ (N. '17) (X, w, l) as above

- $b_+^l := \dim H^+(\hat{X}; l_{\mathbb{R}}) \geq 2$
- $w_2(\hat{X}) + w_2(\hat{k}) + w_1(l_{\mathbb{R}})^2 = 0$
- $\pi^* : H^2(\hat{X}; \mathbb{Z}_2) \longrightarrow H^2(X; \mathbb{Z}_2)$ surjective

$\Longrightarrow$ $\exists$ canonical Spin^c- str $\hat{\mathfrak{s}}_0$ $SW^{Pin}(\hat{X}, \hat{\mathfrak{s}}_0) = \pm 1$

$\exists$ anticanonical —— $\hat{\mathfrak{s}}_0 \otimes \hat{k}$ $SW^{Pin}(\hat{X}, \hat{\mathfrak{s}}_0 \otimes \hat{k}) = \pm 1$

Real Kähler case

$\iota \circlearrowright (X, \omega)$ free antiholo. involution

Consider $\begin{matrix} E \\ \downarrow \mathbb{C} \\ X \end{matrix}$ s.t. $\iota^* E \cong \overline{E}$ ← complex conjugation

$$\begin{array}{ccccccc} I : E & \longrightarrow & \iota^* E \cong \overline{E} & \xrightarrow{\overline{(\cdot)}} & E \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\iota} & X & \xrightarrow{id} & X \end{array}$$

Assume $\boxed{I^2 = id}$

$\begin{matrix} \hat{E} := E/_I \\ \downarrow \\ \hat{X} = X/_\iota \end{matrix}$ $\rightsquigarrow$ Can define $\hat{\mathcal{J}}_0 \hat{\otimes} \hat{E}$

Note $\det \hat{E} \cong \ell_{\mathbb{R}}$

$\boxed{\text{Thm}}$ (N. '17) (X, ω, I) as above

Moduli sp. of $\mathrm{Pin}^-(2)$-monopole solutions on $\hat{S}_0 \hat{\otimes} \hat{E}$

$$\cong \left\{ (\delta, \alpha) \;\middle|\; \begin{array}{l} \delta : \text{holo. str. on } E \quad K \otimes E^{-1} \\ \alpha : \text{nonzero holo. section} \end{array} \right\}^I \Big/ \mathcal{G}_{\mathbb{C}}^I$$

I-invariant part

if $(2c_1(E) - c_1(K))[\omega] < 0$

> 0

$$\cong \left\{ \begin{array}{l} \text{effective divisors} \\ \qquad D \end{array} \text{ s.t. } \begin{array}{l} \cdot\ \mathcal{L}_D \cong E \quad K \otimes E^{-1} \\ \cdot\ \boxed{I \cdot D = D} \end{array} \right\}$$

$D = \sum n_i D_i$ $\quad I \cdot D = \sum n_i I(D_i)$

$\boxed{\text{Thm}}$ (N.'in)

$\iota \curvearrowright (X,\omega)$ as above X: minimal general type $b_+ > 3$

$$SW^{Pin}(\hat{X},\hat{\mathfrak{s}}) = \begin{cases} \pm 1 & \hat{\mathfrak{s}} = \hat{\mathfrak{s}}_0 \text{ or } \hat{\mathfrak{s}}_0 \otimes \hat{K} \\ 0 & \text{otherwise} \end{cases}$$

$\underline{\text{Cf.}}$ [Shuguang Wang '95]

$\iota \curvearrowright (X,\omega)$ as above $\hat{X} = X/\iota$

$$SW^{U(1)}(\hat{X},\mathfrak{s}) = 0 \text{ for } \forall \text{Spin}^c \text{ str } \mathfrak{s}.$$

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- Spin^{c-} structure & $\mathrm{Pin}^-(2)$-monopole eqns
- Canonical Spin^{c-} str.
- Outline of Proofs
- Examples

$Spin^{c-}(4)$

$Pin^{-}(2) = \langle U(1), j \rangle = U(1) \cup jU(1) \subset Sp(1) \subset \mathbb{H}$

- $Pin^{-}(2) \xrightarrow{2:1} O(2)$

$z \in U(1) \subset Pin^{-}(2) \mapsto z^2 \in U(1) \cong SO(2) \subset O(2)$

$j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Def $\boxed{Spin^{c-}(4) := \dfrac{Spin(4) \times Pin^{-}(2)}{\pm 1}}$

- $Spin^{c-}(4) / Pin^{-}(2) = Spin(4)/\pm 1 = SO(4)$
- $Spin^{c-}(4) / Spin(4) = O(2)$
- The id compo of $Spin^{c-}(4) = \dfrac{Spin(4) \times U(1)}{\pm 1} = Spin^{c}(4)$

$Spin^{c-}(4) / Spin^{c}(4) = \{\pm 1\}$

Spin^{c-} structure

$\hat{X}$: oriented connected Riemannian 4-mfd

$Fr(\hat{X})$ SO(4) frame bundle

$X \xrightarrow{2:1} \hat{X}$ (nontrivial) double cover, $\ell := X \times_{\{\pm 1\}} \mathbb{Z}$

[Furuta '08] Spin^{c-} str on $X \to \hat{X}$

$$\left(\begin{array}{ccc} P & P/_{Spin^c(4)} \cong X & P/_{Pin^-(2)} \cong Fr(\hat{X}) \\ \downarrow Spin^{c-}(4) & \downarrow 2:1 & \downarrow SO(4) \\ \hat{X} \quad , & \hat{X} \quad , & \hat{X} \end{array} \right)$$

$\hat{L} := P/_{Spin(4)} \xrightarrow{O(2)} \hat{X}$: characteristic $O(2)$-bdl $\leftrightarrow$ determinat line bdl in Spin^c

ℓ-coefficient Euler class $\tilde{c}_1(\hat{L}) \in H^2(\hat{X}; \ell)$

$$\begin{array}{ccc} P & \xrightarrow{\mathrm{Spin}^c(4)} & P/\mathrm{Spin}^c(4) \cong X \circlearrowleft \iota \quad \iota^2 = 1 \\ {\scriptstyle \mathrm{Spin}^{c-}(4)}\downarrow & \swarrow {\scriptstyle 2:1} & \\ \hat{X} & & \end{array}$$

$J := [1, j] \in \dfrac{\mathrm{Spin}(4) \times \mathrm{Pin}^-(2)}{\pm 1} = \mathrm{Spin}^{c-}(4)$. $\boxed{J^2 = -1}$

· J covers ι

↳ defines a Spin^c str. on X

⇝ $S^\pm$: Spinor b'dls

L : det line

$$S^\pm = P \times_{\mathrm{Spin}^c(4)} \mathbb{H}_\pm \circlearrowleft [J, j] =: I$$

$S^\pm \downarrow X$

$\mathbb{H}_\pm$ ↑ Spin rep.

$\boxed{I^2 = 1,\ I : \text{antilinear}}$

$\hat{S}^\pm := S^\pm / I$ ← Spinor b'dls for the Spin^{c-} str.

$\hat{S}^\pm \downarrow \hat{X}$

Not complex b'dls

$L \circlearrowleft I : L \to \iota^* L \cong \bar{L} \xrightarrow{\overline{(\cdot)}} L$ $\quad \hat{L} = L/I$

$L \downarrow X$

- Twisted Clifford multiplication

$$\rho : T^*\hat{X} \otimes (\ell \otimes i\mathbb{R}) \longrightarrow \mathrm{End}(\hat{S}^+ \oplus \hat{S}^-)$$

- $O(2)$-connection $\hat{A}$ on $\hat{L}$ + Levi-Civita $\rightsquigarrow$ Dirac operator

$$D_{\hat{A}} : \Gamma(\hat{S}^+) \longrightarrow \Gamma(\hat{S}^-) \quad \text{over } \hat{X}$$

- $\pi^*\hat{A}$ on $\pi^*\hat{L}$ $\rightsquigarrow$ $U(1)$-reduction A on L $\rightsquigarrow$ $D_{\hat{A}}$ can be identified with

$$D_A : \Gamma(S^+)^I \longrightarrow \Gamma(S^-)^I \quad \text{over } X$$

$$\boxed{I \cdot A = A}$$

For a $U(1)$-conn. B on L

$\iota^* B$: $U(1)$-conn. on $\bar{L}$

$\overline{\iota^* B}$: ——— on L

$$I \cdot B := \overline{\iota^* B}$$

$\mathrm{Pin}^-(2)$-monopole equations

$$\begin{cases} D_{\hat{A}}\hat{\Phi} = 0 \\ \rho(F_{\hat{A}}^+) = q(\hat{\Phi}) \end{cases}$$

$\hat{A}$: $O(2)$-conn. on $\hat{L}$

$\hat{\Phi} \in \Gamma(\hat{S}^+)$ $F_{\hat{A}}^+ \in \Omega^+(\ell \otimes i\mathbb{R})$

$q(\hat{\Phi}) =$ "$(\hat{\Phi} \otimes \hat{\Phi}^*)_0$" $\in \mathrm{End}(\hat{S}^+)$

$$\boxed{\mathrm{Pin}^-(2)\text{-monopole} = I\text{-inv. SW}}$$

$\mathrm{Pin}^-(2)$ solution $\longleftrightarrow$ I-inv. SW solution

$(\hat{A}, \hat{\Phi}) \longmapsto (A, \Phi) = (\pi^*\hat{A}, \pi^*\hat{\Phi})$

$I(A, \Phi) = (A, \Phi)$

$$\boxed{I = \overline{\tau^*(\cdot)}}$$

Gauge Symmetry

$$\mathcal{G}_{Pin} = \Gamma\left(\begin{array}{c} X \times_{\{\pm1\}} U(1) \\ \downarrow \\ \hat{X} \end{array}\right) \cong \mathcal{G}_{U(1)}^{I}$$

$\{\pm 1\} \curvearrowright U(1) \quad z \mapsto z^{-1}$

$\mathcal{G}_{U(1)} = \mathrm{Map}(X, U(1))$

$\circlearrowleft$

$I : f \mapsto \overline{\iota^* f}$

Moduli space

$$\mathcal{M}_{Pin} := \{Pin^-(2) \text{ solutions}\}/\mathcal{G}_{Pin} \cong \mathcal{M}_{U(1)}^{I} = \{I\text{-inv. SW sol on } X\}^{I}/\mathcal{G}_{U(1)}^{I}$$

- compact
- may be non-orientable

Pin$^-$(2)-monopole inv.

$$\dim \mathcal{M}_{Pin} = 0 \Rightarrow SW^{Pin}(X, \mathfrak{s}) = \# \mathcal{M}_{Pin} \begin{array}{ll} \in \mathbb{Z}_2 & \text{nonori} \\ \in \mathbb{Z} & \text{ori} \end{array}$$

Canonical Spin^c str.

(X, J) almost complex 4-mfd, Riemannian

- $\begin{array}{c} Fr(X) \\ \downarrow SO(4) \\ X \end{array} \xrightarrow{\text{reduction}} \begin{array}{c} F \\ \downarrow U(2) \\ X \end{array}$

- $U(2) = \dfrac{U(1) \times SU(2)}{\pm 1} \hookrightarrow \dfrac{SU(2) \times SU(2) \times U(1)}{\pm 1} = Spin^c(4)$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (z, q) & \longmapsto & (z, q, z) \end{array}$$

$\rightsquigarrow$ canonical Spin^c structure

s.t. $S^+ = \underline{\mathbb{C}} \oplus K^{-1}$

Canonical $\boxed{\mathrm{Spin}^{c-}}$ structure

$$\hat{U}(2) := \frac{\boxed{Pin^-(2)} \times SU(2)}{\pm 1} \subset \frac{\overset{\overset{Sp(1)}{\shortparallel}}{SU(2)} \times SU(2) \times Pin^-(2)}{\pm 1} = \mathrm{Spin}^{c-}(4)$$

$$(z, q) \longmapsto (z, q, z)$$

- a $\hat{U}(2)$-reduction of $Fr(\tilde{X})$ defines a Spin^{c-}-str.

$\underline{\underline{Q}}$ When $\exists\, \hat{U}(2)$-reduction?

Recall our setting

$\iota \curvearrowright (X, \omega)$

anti-holo. free, $\iota^2 = \mathrm{id}$

$$\begin{array}{c} X \\ \downarrow \pi \\ \hat{X} = X/\iota \end{array} \qquad \begin{array}{l} \ell = X \times_{\{\pm 1\}} \mathbb{Z} \\ \\ \ell_{\mathbb{R}} = \ell \otimes \mathbb{R} \end{array}$$

$$\begin{array}{ccc} Fr(X) & & F \\ \downarrow SO(4) & \overset{\text{reduction}}{\rightsquigarrow} & \downarrow U(2) \\ X & & X \end{array}$$

$I \curvearrowright K^{-1}$

$\hat{K}^{-1} = K^{-1}/I$

<u>Prop.</u> $\begin{cases} w_2(\hat{X}) + w_2(\hat{K}) + w_1(\ell_{\mathbb{R}})^2 = 0 \\ \pi^* : H^2(\hat{X}; \mathbb{Z}_2) \to H^2(X; \mathbb{Z}_2) \text{ surjective} \end{cases}$

$\Rightarrow \exists$ reduction $\begin{array}{c} \hat{F} \\ \downarrow \hat{U}(2) \\ \hat{X} \end{array}$ of $Fr(\hat{X})$ s.t. $\cdot\ \pi^* \hat{F} \overset{\text{red}}{\rightsquigarrow} F$

$\cdot$ its Spin^{c-} str has

$\hat{S}^+ = (\underline{\mathbb{R}} \oplus \ell_{\mathbb{R}}) \oplus \hat{K}^{-1}$

$\longrightarrow$ canonical Spin^{c-} str.

Note $\pi^* \hat{S}^+ \cong \underline{\mathbb{C}} \oplus K^{-1}$

- $\{\mathrm{Spin}^{c-}\text{ strs}\}/_{\mathrm{iso}} \circlearrowleft H^2(\hat{X};\ell)$

$$= \{O(2)\text{-bdls } \hat{E} \text{ s.t. } \det \hat{E} = \ell_{\mathbb{R}}\}/_{\mathrm{iso}}$$

$$\hat{\mathfrak{s}}_0 \mapsto \hat{\mathfrak{s}}_0 \hat{\otimes} \hat{E}$$

Cf $\{\mathrm{Spin}^{c}\text{ strs}\}/_{\mathrm{iso}} \circlearrowleft H^2(X;\mathbb{Z})$

$$= \{U(1)\text{ bdls } E\}/_{\mathrm{iso}}$$

$$\mathfrak{s}_0 \mapsto \mathfrak{s}_0 \otimes E$$

- $\hat{\mathfrak{s}}_0 \hat{\otimes} \hat{K}$: anti canonical Spin^{c-} str

Outline of proofs of $\boxed{\text{Thm}}$s

$\iota \circlearrowright (X, \omega, J)$: symplectic or Kähler
anti holo.

(almost) cpx str $\begin{cases} \text{is not preserved by } \iota. \\ \text{does not descend to } \hat{X}. \end{cases}$

Basic Strategy

Objects on $\hat{X}$ $\longleftrightarrow$ I-invariant objects on X

$$I = \overline{\iota^{*}(\cdot)}$$

Real symplectic case

$\exists \hat{\omega} \in \Omega^2(\hat{X}; l_{\mathbb{R}})$ s.t. $\pi^* \hat{\omega} = \omega$

$\hat{\mathcal{S}}_0 \rightsquigarrow \hat{S}^+ = (\underline{\mathbb{R}} \oplus l_{\mathbb{R}}) \oplus \hat{K}^{-1}$ $\quad \hat{u}_0 := (1,0) \oplus 0 \in \Gamma(\hat{S}^+)$

Prop. $\exists !$ $\hat{A}_0$: $O(2)$-conn. on $\hat{K}^{-1}$ cf [Taubes]

up to gauge

$$D_{\hat{A}_0} \hat{u}_0 = 0$$

Perturb the $\mathrm{Pin}^-(2)$-monopole eqn by $tr\hat{\omega}$ $(r \gg 0)$

$(\hat{A}_0, \hat{u}_0)$ is a $\mathrm{Pin}^-(2)$-monopole solution for $\forall r$.

$\Big\{$ lift

(A_0, u_0) unique (I-inv) SW sol. $\rightsquigarrow$ $\boxed{SW^{\mathrm{Pin}}(\hat{\mathcal{S}}_0) = \pm 1}$

for $tr\omega$

Real Kähler case

$$\iota \curvearrowright (X, \omega)$$

$$\Omega^{p,q} \xrightarrow{\iota^*} \Omega^{q,p} \xrightarrow{\overline{(\cdot)}} \Omega^{p,q} \qquad I = \overline{\iota^*(\cdot)} \curvearrowright \Omega^{p,q}$$

$$\boxed{I^2 = id}$$

Consider

$\hat{S}_0 \otimes \hat{E}$ where $\begin{matrix} \hat{E} \\ \downarrow \\ \hat{X} \end{matrix}$: $\mathbb{R}^2$-bdl s.t. $\det \hat{E} = \ell_{\mathbb{R}}$

$E := \pi^* \hat{E}$ $\quad E$ admits a str of $\mathbb{C}$-line bdl s.t. $\iota^* E \cong \overline{E}$

with

$$\begin{tikzcd} E \arrow[r, "\iota^*"] \arrow[rrr, bend right, "I"'] & \iota^* E \cong \overline{E} \arrow[rr, "\overline{(\cdot)}"] & & E \end{tikzcd}$$

$$\boxed{I^2 = id}$$

$\pi^*(\hat{S}_0 \hat{\otimes} \hat{E})$ has a canonical reduction to the Spin^c-str

$$\boxed{S_0 \otimes E} \qquad S^+ = E \oplus E \otimes K^{-1}$$

SW eqn on $S_0 \otimes E$ can be written as

$\bar{\partial}_B \alpha + \bar{\partial}_B^* \beta = 0$

B : $U(1)$-conn. on E

$\alpha \in \Omega^{00}(E)$

$2 \bar{F}_B^{02} - \frac{1}{2} \beta \bar{\alpha} = 0$

$\beta \in \Omega^{02}(E)$

I

$\left\{ \Lambda F_B - \frac{i}{2} s_g + \frac{i}{8} (|\beta|^2 - |\alpha|^2) \right\} \omega = 0$

$\overline{\tau^* B}$: $U(1)$-conn. on E $\qquad I \cdot B := \overline{\tau^* B}$

$$\Omega^{p,q}(E) \xrightarrow{\tau^*} \Omega^{q,p}(\bar{E}) \xrightarrow{\overline{(\cdot)}} \Omega^{p,q}(E)$$

I

$$\{\mathrm{Pin}^-(2)\text{-monopole sol.}\}/\mathcal{G}_{\mathrm{Pin}} \cong \{I\text{-inv. SW sol}\}/\mathcal{G}^{I}_{U(1)}$$

Prop ① $(2c_1(E)-c_1(K))[\omega]<0 \Rightarrow \beta=0$

② $(2c_1(E)-c_1(K))[\omega]>0 \Rightarrow \alpha=0$

Case① SW $\Leftrightarrow$

$$\begin{cases}\bar\partial_B\alpha=0\\ F_B^{0,2}=0 \Rightarrow \bar\partial_B \text{ defines a holo. str. on } E\\ \Lambda F_B-\frac{i}{2}s_g-\frac{i}{8}|\alpha|^2=0\end{cases}$$

$\boxed{\text{Vortex eq}}$

Case② SW $\Leftrightarrow$ Similarly Vortex eq

$\boxed{\text{Thm}}$ $\{\mathrm{Pin}^-(2)\text{-monopole sol.}\}/\mathcal{G}_{\mathrm{Pin}} \cong \{I\text{-inv. SW sol}\}/\mathcal{G}_{U(1)}^I$

$\cong \{I\text{-inv Vortex sol}\}/\mathcal{G}_{U(1)}^I$

$\mathcal{G}_{\mathbb{C}} = \mathrm{Map}(X, \mathbb{C}^*)$

$\circlearrowleft$

$I \quad f \longmapsto \overline{I^* f}$

$\cong \left\{ (\delta, \alpha) \;\middle|\; \begin{array}{l} \delta : \text{holo str.} \\ \delta\alpha = 0,\ \alpha \neq 0 \end{array} \right\}^I / \mathcal{G}_{\mathbb{C}}^I$

$\cong \{\text{effective divisors}\}^I$

Example 1

$X_{4k} \subset \mathbb{CP}^3 \circlearrowleft \iota \quad [x_0, x_1, x_2, x_3] \mapsto [\overline{x_1}, -\overline{x_0}, \overline{x_3}, -\overline{x_2}]$

defined by a $\boxed{\text{real}}$ polynomial of deg $= 4k$

Prop. $\exists$ canonical Spin^c- $\hat{\mathfrak{s}}_0$ on $\hat{X}_{4k} = X_{4k}/\iota$

$\boxed{k=1}$ X_4 : K3 $\qquad \hat{X}_4 = X_4/\iota$: Enriques

$$SW^{Pin}(\hat{\mathfrak{s}}) = \begin{cases} \pm 1 & \hat{\mathfrak{s}} = \hat{\mathfrak{s}}_0 \ (\cong \hat{\mathfrak{s}}_0 \hat{\otimes} \hat{K}) \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{k>1}$ X_{4k} : general type

$$SW^{Pin}(\hat{\mathfrak{s}}) = \begin{cases} \pm 1 & \hat{\mathfrak{s}} = \hat{\mathfrak{s}}_0 \text{ or } \hat{\mathfrak{s}}_0 \hat{\otimes} \hat{K} \\ 0 & \text{otherwise} \end{cases}$$

Example 2 Elliptic surfaces

$$E(n) = \underbrace{E(1) \#_f \cdots \#_f E(1)}_{n}$$

$$\varpi \downarrow$$

$$\mathbb{C}P^1$$

$E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$

$\#_f$: fiber sum

Prop. $\exists$ anti-holo. free inv $\tau \curvearrowright E(2n)$ covering the antipodal map on $\mathbb{C}P^1 = S^2$.

- $E(2n)$ admits a Kähler form ω s.t $\boxed{\tau^* \omega = -\omega}$
- $\exists$ canonical Spin^{c-} str $\hat{\mathfrak{s}}_0$ on $\widehat{E(4m+2)} = E(4m+2)/_2$

Choose general fibers $F_1, \dots, F_k$

I-inv. divisor $D_k := \sum_{i=1}^{k} (F_i + I F_i) \longleftrightarrow$ the corresponding line bdl E_k

We can see $I \curvearrowright E_k$. $\hat{E}_k := E_k / I$

$\boxed{\text{Want to calculate } SW^{Pin}(\hat{\mathfrak{s}}_0 \hat{\otimes} \hat{E}_k)}$

$$\#\{I\text{-inv divisors } D \text{ s.t. } \mathcal{L}_D \cong E_k\} = \binom{2m}{k}$$

<u>Conj</u> $SW^{Pin}(\hat{\mathfrak{s}}_0 \hat{\otimes} \hat{E}_k) = \pm \binom{2m}{k}$

Orientation is very very subtle!!

Problems

① Generalization to the case when ι is NOT free

$\to$ Find applications to real algebraic geometry.

② $SW^{Pin} \underset{??}{=}$ real Gr Cf [Tian - S. Wang]

③ $Pin^-(2)$-version of the Witten Conjecture

$SW(U(1)) \longleftrightarrow$ $SU(2)$ / $SO(3)$ instanton

$Pin^-(2)$-monopole $\underset{??}{\longleftrightarrow}$ $Pin(3)$ / $O(3)$ instanton